2016

Some local estimates and a uniqueness result for the entire biharmonic heat equation

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Publication Details

Simon, M. & Wheeler, G. (2016). Some local estimates and a uniqueness result for the entire biharmonic heat equation. Advances in Calculus of Variations, 9 (1), 77-99.

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Abstract

We consider smooth solutions to the biharmonic heat equation on ℝn x [0,T] for which the square of the Laplacian at time t is globally bounded from above by k0/t for some k0 in ℝ+, for all t ∈ [0,T]. We prove local, in space and time, estimates for such solutions. We explain how these estimates imply uniqueness of smooth solutions in this class.

Disciplines

Engineering | Science and Technology Studies

Publication Details

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SOME LOCAL ESTIMATES AND A UNIQUENESS RESULT FOR THE ENTIRE BIHARMONIC HEAT EQUATION

MILES SIMON AND GLEN WHEELER

Abstract. We consider smooth solutions to the biharmonic heat equation on $\mathbb{R}^n \times [0,T]$ for which the square of the Laplacian at time t is globally bounded from above by k_0/t for some k_0 in $\mathbb{R}^+$, for all $t \in [0,T]$. We prove local, in space and time, estimates for such solutions. We explain how these estimates imply uniqueness of smooth solutions in this class.

1. Introduction

In this paper we prove local in space and time estimates for solutions $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ of the biharmonic heat flow,

$$\frac{\partial}{\partial t} u = -\Delta^2 u, \tag{1.1}$$

assuming that we have some global in time control on how the solution behaves as $t \searrow 0$. This control takes the form

$$t|\Delta u|^2(x,t) \leq k_0 < \infty \tag{1.2}$$

for $t \in [0,T]$ for all $x \in \mathbb{R}^n$ for some fixed $k_0 \in \mathbb{R}^+$. This means that it is possible for $|\Delta u|^2(x,t)$ to approach infinity as $t \searrow 0$, but if it does so, then we have some control over the rate at which this occurs. Here Δu refers to the spatial Lapacian of u, $\Delta u(x,t) = \sum_{i=1}^n \nabla_i \nabla_i u(x,t)$ where $\nabla_i u(x,t)$ is the partial derivative of u with respect to x_i.

The growth condition (1.2) is natural in the following senses. It is scale invariant: see the explanation of the scale invariance of b just after the the definition of (A1) in Section 3. This behaviour also does occur in an asymptotic sense. That is, it is possible to construct a solution $u \in C^\infty(\mathbb{R}^n \times (0,T))$ and to find points $x(t) \in \mathbb{R}^n$ for all $t > 0$ such that $|\Delta u|^2(x(t),t) = \frac{k_0}{t}$ for some fixed $k_0 > 0$ for all $t > 0$. We also find points $y(t) \in \mathbb{R}^n$ for all $t > 0$ such that $(\Delta^2 u)(y(t),t) = \frac{k_0}{t}$ for some fixed $k_0 \neq 0$ for all $t > 0$. That is, the speed of u is not integrable in time. See the example in Section 6.

Our first result is the following local estimate.

Theorem 1.1. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $T < \infty$, be a smooth solution to* (1.1) *that satisfies*

$$|\Delta u|^2(x,t) \leq \frac{k_0}{t} \tag{1.3}$$

Date: February 2013.

2010 *Mathematics Subject Classification.* 35B45, 35K25, 35K35.

Key words and phrases. fourth-order parabolic partial differential equations, heat flow, local estimates, uniqueness.

for some $k_0 \in \mathbb{R}$*, for all* $t \in [0,T]$*, all* $x \in \mathbb{R}^n$ *and*

$$\sup_{B_1(0)} \sum_{i=0}^{2n+2} |\nabla^i u_0|^2 \le k_1 < \infty,$$

for some $k_1 \in \mathbb{R}$*, where* $u_0(\cdot) := u(\cdot, 0)$*. Then there exists an* $N = N(n, k_0, k_1) > 0$ *such that*

$$|\Delta u|^2(x,t) \le \frac{N}{(1-|x|)^4}$$

for all x*,* t *which satisfy* $x \in \overline{B}_1(0)$*,* $(1-|x|)^4 \ge Nt$*, and* $t \le \frac{1}{N}$ *,* $t \le T$*.*

For $i \in \mathbb{N}$ in the above and all that follows, $\nabla^i u(x,t)$ refers to the full i-th spatial derivative, and $|\nabla^i u(x,t)|$ the standard norm thereof. For example: $\nabla^2 u(x,t) = (\nabla_i \nabla_j u(x,t))_{i,j \in \{1,\dots,n\}}$ and $|\nabla^2 u(x,t)|^2 = \sum_{i,j=1}^n |\nabla_i \nabla_j u(x,t)|^2$. The operator Δ^k is defined by $\Delta^k = (\Delta)^k$.

If we have control on other derivatives as $t \searrow 0$ then we obtain results on higher regularity.

Theorem 1.2. *Let* $s \in \mathbb{N}$*,* $s \ge 2$ *and* $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$*,* $T < \infty$ *be a smooth solution to* (1.1) *which satisfies*

$$\left(|\nabla^s u| + |\nabla^{s-1} u|^{s/(s-1)} + \ldots + |\nabla u|^s\right)(x,t) \le \frac{k_0}{t^{s/4}} \tag{1.4}$$

for some $k_0 \in \mathbb{R}$*, for all* $t \in [0,T]$*, all* $x \in \mathbb{R}^n$ *and*

$$\sup_{B_1(0)} \sum_{i=0}^{2n+s+1} |\nabla^i u_0|^2 \le k_1 < \infty,$$

for some $k_1 \in \mathbb{R}$*. Then there exists an* $N = N(n, k_0, k_1, s) > 0$ *such that*

$$\left(|\nabla^s u| + |\nabla^{s-1} u|^{s/(s-1)} + \ldots + |\nabla u|^s\right)(x,t) \le \frac{N}{(1-|x|)^s}$$

for all x*,* t *which satisfy* $x \in \overline{B}_1(0)$*,* $(1-|x|)^4 \ge Nt$*, and* $t \le \frac{1}{N}$*,* $t \le T$*.*

In Section 5 we construct an example of a solution to (1.1) which starts off indentically equal to zero, becomes immediately non-zero and is smooth in space and time. Solutions of this type for the heat equation are known to exist and were constructed by Tychonoff, see [T]. In particular, smooth solutions are not uniquely determined by their initial values: $u(\cdot,\cdot) = 0$ is also a solution. If however we consider smooth solutions which satisfy (1.3) then the solution is uniquely determined by it's initial value, as we show in Section 4.2. The theorem we prove is:

Theorem 1.3. *Let* $u, v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$*,* $T < \infty$ *be smooth solutions to* (1.1) *which satisfy*

$$|\Delta v|^2(x,t) + |\Delta u|^2(x,t) \le \frac{k_0}{t}$$

for some $k_0 \in \mathbb{R}$*, for all* $t \in [0,T]$ *and all* $x \in \mathbb{R}^n$ *and*

$$u_0(\cdot) = v_0(\cdot).$$

Then $u \equiv v$*.*

The uniqueness problem for the classical heat flow has a rich history. In the setting where one assumes the solution is non-negative, D. Widder established uniqueness for the heat flow on $\mathbb{R}$ for solutions whose initial value is zero, see Theorem 5 of [W]. His method relied upon a specific integral representation of the solution, Theorem 4 of [W], which is valid for non-negative solutions.

This proved to be readily generalisable and so influential as to be given its own name: a uniqness theorem is of Widder-type if the only hypotheses are on the geometry of the ambient space and that the solution be non-negative. For example, Aronson [Ar] proved that non-negative solutions to second order linear equations of divergence form (the coefficents of the operator being sufficiently regular) in $\mathbb{R}^n$ are uniquely determined by their initial data: see Section 5 of that paper.

Here we take an approach reminiscent of [Si] that complements the existing literature. Our assumptions for Theorem 1.3 are global but we do not require any non-negativity of the solution. Although the flow (1.1) is higher-order, we are able to obtain our estimates using pointwise assumptions, as opposed to integral conditions.

The paper is organised as follows. Section 3 contains the proof of Theorem 1.1 and Theorem 1.2. These proofs require some energy estimates for solutions to (1.1), which is the subject of Section 2. Section 4 contains the proof of Theorem 1.3, and Section 5 contains full details on the Tychonoff-type solutions discussed above. We present in Section 6 details on the construction of the example mentioned above which shows that control of the form (1.2) is natural. We also show that the solution in this example has a speed which is not integrable.

Some of the estimates from Section 3 and Section 2 rely on interpolation inequalities which are not readily available in the current literature. These interpolation inequalities are proved in the Appendix.

2. A priori Energy Estimates

In this section we shall prove some estimates for the weighted energies

$$E^k_\eta(u) = \int_{\mathbb{R}^n} |\Delta^k u|^2\,\eta, \quad \eta \in C^\infty_{loc}(\mathbb{R}^n), \quad \text{supp}\,\eta \subset\subset \mathbb{R}^n, \tag{2.1}$$

where $k \in \mathbb{N}_0$ and $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a smooth solution to (1.1). In the above equation and in that which follows all integrals are with respect to Lebesgue measure. Note that E^k_η are all finite for any $k \in \mathbb{N}_0$ and $t \in [0,T]$ since $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is smooth. The purpose of the a priori estimates in this section is to quantify how global quantities such as the various Sobolev norms of the solution behave along the flow (1.1).

Lemma 2.1. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a smooth solution to* (1.1). *For all $t \in [0,T]$,*

$$\begin{aligned}&\frac{d}{dt}E^k_\eta(u) + 2E^{k+1}_\eta(u)\\ &\quad = -2\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta) + 2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta)\,.\end{aligned}\tag{2.2}$$

for all $k \in \mathbb{N}_0$.

Proof. Differentiating,

$$\begin{aligned}
\frac{d}{dt}E^k_\eta(u) &= 2\int_{\mathbb{R}^n}(\Delta^k u)(-\Delta^{k+2}u)\,\eta \\
&= 2\int_{\mathbb{R}^n}(\nabla_i\Delta^k u)(\nabla_i\Delta^{k+1}u)\,\eta \,+ 2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta) \\
&= -2\int_{\mathbb{R}^n}|\Delta^{k+1}u|^2\,\eta \,- 2\int_{\mathbb{R}^n}(\nabla_i\Delta^k u)(\nabla_i\eta)(\Delta^{k+1}u) \\
&\qquad + 2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta) \\
&= -2E^{k+1}_\eta(u) \\
&\qquad - 2\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta) \,+ 2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta)\,.
\end{aligned}$$

Rearranging gives the lemma. □

We now specialise by setting $\eta = \gamma^s$, $s > 0$ to be chosen, and $\gamma \in C^\infty_{\text{loc}}(\mathbb{R}^n)$ satisfying

$$(\gamma) \qquad \chi_{B_\rho(0)} \le \gamma \le \chi_{B_{2\rho}(0)}, \quad \rho > 0, \quad \text{and} \quad |\nabla\gamma| \le \frac{c_\gamma}{\rho}, \quad |\nabla^2\gamma| \le \frac{c_\gamma}{\rho^2}\,,$$

where $c_\gamma \ge 1$ is an absolute constant depending only on n.

In the following proofs we make extensive use of the elementary inequality

$$ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2 \tag{2.3}$$

for a, b real numbers and $\varepsilon > 0$.

Lemma 2.2. *Let $u \in C^\infty_{loc}(\mathbb{R}^n)$. Suppose $\eta = \gamma^s$, $s > 8$, and γ, c_γ are as in (γ). Then for any $\varepsilon > 0$ and for all $k \in \mathbb{N}$ we have*

$$-2\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta) \,\le \varepsilon E^{k+1}_\eta(u) + \frac{c_1(\varepsilon,s,n)}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\,,$$

where $c_1(\varepsilon, s, n) < \infty$ is a constant depending on ε, s and n.

Proof. Throughout the proof δ_i denote positive parameters to be chosen. Using (2.3) and (γ),

$$\begin{aligned}
-2\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta) \,&= -2s\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\gamma)\gamma^{s-1} \\
&\le \delta_1 E^{k+1}_\eta(u) + \frac{c_\gamma^2 s^2}{\delta_1\rho^2}\int_{\mathbb{R}^n}|\nabla\Delta^k u|^2\gamma^{s-2}\,.
\end{aligned} \tag{2.4}$$

Now,

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla\Delta^k u|^2\gamma^{s-2} \\
&\quad = -\int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k+1}u)\gamma^{s-2} - (s-2)\int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^k u)(\nabla_i\gamma)\gamma^{s-3} \\
&\quad \le \rho^2\delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{1}{4\delta_2\rho^2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\Delta^k u|^2\gamma^{s-2} + \frac{c_\gamma^2(s-2)^2}{2\rho^2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \\
&\quad = \rho^2\delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\Delta^k u|^2\gamma^{s-2} \\
&\qquad \frac{1}{2\rho^2}(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2)\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4}
\end{aligned}
$$

Absorbing the second term on the right into the left we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla\Delta^k u|^2\gamma^{s-2} \\
&\quad \le 2\rho^2\delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{1}{\rho^2}\Big(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\Big)\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4}\,.
\end{aligned}
\tag{2.5}
$$

We now need an interpolation inequality. From Lemma 7.2 we know that

$$
\begin{aligned}
&\frac{1}{\rho^2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \\
&\quad \le \delta_3\rho^2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{c_{\delta_3}}{\rho^6}\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,,
\end{aligned}
$$

where $c_{\delta_3} = c_{\delta_3}(\delta_3, s, n) = 2^4\big(\frac{1}{\delta_3} + 2^9 s^4 c_\gamma^4\big)$. Using this in (2.5) we obtain

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\Delta^k u|^2\gamma^{s-2} &\le 2\rho^2\delta_2 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s \\
&\quad + \left(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\right)\left(\delta_3\rho^2\int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{c_{\delta_3}}{\rho^6}\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\right) \\
&= \left(2\rho^2\delta_2 + \delta_3\rho^2\left(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\right)\right)\int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s \\
&\quad + \frac{c_{\delta_3}}{\rho^6}\left(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\right)\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\frac{1}{\rho^2}\int_{\mathbb{R}^n} |\nabla\Delta^k u|^2\gamma^{s-2} &\le \left(\delta_3\Big(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\Big) + 2\delta_2\right)\int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s \\
&\quad + \frac{c_{\delta_3}}{\rho^8}\Big(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\Big)\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,.
\end{aligned}
\tag{2.6}
$$

Combining (2.6) with (2.4) gives

$$
\begin{aligned}
&-2\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta)\\
&\qquad\le\Big[\delta_1+\frac{c_\gamma^2 s^2}{\delta_1}\Big(\delta_3\Big(\frac{1}{2\delta_2}+c_\gamma^2(s-2)^2\Big)+2\delta_2\Big)\Big]E_\eta^{k+1}(u)\\
&\qquad\quad+\frac{c_\gamma^2 s^2}{\delta_1}\frac{c_{\delta_3}}{\rho^8}\Big(\frac{1}{2\delta_2}+c_\gamma^2(s-2)^2\Big)\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\,.
\end{aligned}
$$

We choose $\delta_i=\delta_i(s,\varepsilon,n)>0$ so that

$$
\Big[\delta_1+\frac{c_\gamma^2 s^2}{\delta_1}\Big(\delta_3\Big(\frac{1}{2\delta_2}+c_\gamma^2(s-2)^2\Big)+2\delta_2\Big)\Big]\le\varepsilon\,.
$$

This can be achieved by the choice

$$
\delta_1=\frac{\varepsilon}{4},\quad \delta_2=\frac{\varepsilon^2}{32c_\gamma^2 s^2},\quad \delta_3=\frac{\varepsilon^4}{8c_\gamma^4 s^2(16s^2+\varepsilon^2(s-2)^2)}
$$

for example. Recalling the definition of $c_{\delta_3}=c(\delta_3,s,n)=2^4\big(\frac{1}{\delta_3}+2^9s^4c_\gamma^4\big)$ yields the result. $\square$

Lemma 2.3. *Let $u\in C^\infty_{loc}(\mathbb{R}^n)$. Suppose $\eta=\gamma^s$, $s>8$, and γ, c_γ are as in (γ). Then for any $\varepsilon>0$ and for all $k\in\mathbb{N}$ we have*

$$
2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta)\ \le\varepsilon E_\eta^{k+1}(u)+\frac{c_2(\varepsilon,s,n)}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\,,
$$

where $c_2(\varepsilon,s,n)<\infty$ is a constant depending on ε,s,n.

Proof. Again, throughout the proof $\delta_i>0$ denote positive parameters to be chosen. Integrating by parts,

$$
\begin{aligned}
2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta)\ &=-2\int_{\mathbb{R}^n}(\nabla_i\Delta^k u)(\Delta^{k+1}u)(\nabla_i\eta)\\
&\quad-2\int_{\mathbb{R}^n}(\Delta^k u)(\Delta^{k+1}u)(\Delta\eta)
\end{aligned}\tag{2.7}
$$

Lemma 2.2 deals with the first term:

$$
-2\int_{\mathbb{R}^n}(\nabla_i\Delta^k u)(\Delta^{k+1}u)(\nabla_i\eta)\ \le\frac{\varepsilon}{2}E_\eta^{k+1}(u)+\frac{c_1(\frac{\varepsilon}{2},s,n)}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\,.\tag{2.8}
$$

Since $\Delta\eta=s\gamma^{s-1}\Delta\gamma+s(s-1)\gamma^{s-2}|\nabla\gamma|^2$ we have

$$
\begin{aligned}
&-2\int_{\mathbb{R}^n}(\Delta^k u)(\Delta^{k+1}u)(\Delta\eta)\\
&\qquad\le 2\delta_1\int_{\mathbb{R}^n}|\Delta^{k+1}u|^2\eta+\frac{1}{\delta_1\rho^4}\Big(c_\gamma^2 s^2+c_\gamma^4 s^2(s-1)^2\Big)\int_{\mathbb{R}^n}|\Delta^k u|^2\gamma^{s-4}\,,
\end{aligned}\tag{2.9}
$$

where we used the fact that $\gamma^{s-2}(\cdot) \le \gamma^{s-4}(\cdot)$, which is true since $0 \le \gamma(\cdot) \le 1$ on $\mathbb{R}^n$. Lemma 7.2 yields the estimate

$$\begin{aligned}\frac{1}{\delta_1\rho^4}\Big(c_\gamma^2 s^2 + c_\gamma^4 s^2(s-1)^2\Big)&\int_{\mathbb{R}^n}|\Delta^k u|^2\gamma^{s-4}\\ &\le \frac{\delta_2}{\delta_1}\Big(c_\gamma^2 s^2 + c_\gamma^4 s^2(s-1)^2\Big)\int_{\mathbb{R}^n}|\Delta^{k+1} u|^2\gamma^{s}\\ &\quad+\frac{c_{\delta_2}}{\delta_1\rho^8}\Big(c_\gamma^2 s^2 + c_\gamma^4 s^2(s-1)^2\Big)\int_{\mathbb{R}^n}|\Delta^{k-1} u|^2\gamma^{s-8}\end{aligned}$$

where $c_{\delta_2} = c(\delta_2, s, n) = 2^4\big(\frac{1}{\delta_2} + 2^9 s^4 c_\gamma^4\big)$. Combining this with (2.9) we get

$$\begin{aligned}-2\int_{\mathbb{R}^n}(\Delta^k u)(\Delta^{k+1}u)(\Delta\eta) \;&\le \left(2\delta_1 + \frac{\delta_2}{\delta_1}\Big(c_\gamma^2 s^2 + c_\gamma^4 s^2(s-1)^2\Big)\right)\int_{\mathbb{R}^n}|\Delta^{k+1}u|^2\eta\\ &\quad+\frac{c_{\delta_2}}{\delta_1\rho^8}\left(c_\gamma^2 s^2 + c_\gamma^4 s^2(s-1)^2\right)\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\,.\end{aligned}\tag{2.10}$$

Combining (2.10) with (2.7)-(2.8) and choosing $\delta_i = \delta_i(\varepsilon, n, s) > 0$ so that

$$\frac{\delta_2}{\delta_1}\Big(c_\gamma^2 s^2 + c_\gamma^4 s^2(s-1)^2\Big) + 2\delta_1 = \varepsilon/2$$

yields the result. One possible choice is

$$\delta_1 = \frac{\varepsilon}{16}, \quad \delta_2 = \frac{\varepsilon^2}{128c_\gamma^2\Big(s^2 + c_\gamma^2 s^2(s-1)^2\Big)}\,.$$

□

Corollary 2.4. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a smooth solution to (1.1). Suppose $\eta = \gamma^s$, $s > 8$, and γ, c_γ, are as in (γ), and $k \in \mathbb{N}$. Then*

$$\frac{d}{dt}E_\eta^k(u) + \frac{3}{2}E_\eta^{k+1}(u) \le \frac{c_3(n,s)}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}dx\,,$$

where $c_3(n,s)$ is constant depending only on n and s.

Proof. We combine Lemmata 2.1–2.3 as follows. Lemma 2.1 states that

$$\begin{aligned}&\frac{d}{dt}E_\eta^k(u) + 2E_\eta^{k+1}(u)\\ &\quad= -2\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta) \,+ 2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta)\,.\end{aligned}\tag{2.11}$$

The two terms on the right hand side are estimated by Lemma 2.2 and Lemma 2.3 respectively. Adding together the estimates we find, for any $\varepsilon_1, \varepsilon_2 > 0$,

$$\begin{aligned}-2&\int_{\mathbb{R}^n}(\Delta^{k+1}u)(\nabla_i\Delta^k u)(\nabla_i\eta) + 2\int_{\mathbb{R}^n}(\Delta^k u)(\nabla_i\Delta^{k+1}u)(\nabla_i\eta)\\ &\le (\varepsilon_1+\varepsilon_2)E_\eta^{k+1}(u) + \frac{c_1(\varepsilon_1,n,s)+c_2(\varepsilon_2,n,s)}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}.\end{aligned}$$

In particular choosing $\varepsilon_i = \frac{1}{4}$ and combining this with (2.11) we have

$$\frac{d}{dt}E_\eta^k(u) + 2E_\eta^{k+1}(u) \le \frac{1}{2}E_\eta^{k+1}(u) + \frac{c_3}{2\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\,,$$

where c_3 is a constant depending only on s and n. Absorbing the first term on the right into the left yields the claimed estimate. □

Corollary 2.5. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ be a smooth solution to (1.1). Suppose $k \in \mathbb{N}$, $\eta = \gamma^s$, $s > 4k$, where γ, c_γ are as in (γ). Then*

$$\frac{d}{dt}E^k_\eta(u) + E^{k+1}_\eta(u) \le \frac{c_4(n,s)}{\rho^{4k}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}\,,$$

where $c_4(n,s)$ is a constant depending only on n and s.

Proof. We first consider the case where $k = 1$. Using Lemma 2.1 and integration by parts we find

$$\begin{aligned}
&\frac{d}{dt}E^1_\eta(u) + 2E^2_\eta(u)\\
&\quad = -2\int_{\mathbb{R}^n}(\Delta^2 u)(\nabla_i\Delta u)(\nabla_i\eta)\;+2\int_{\mathbb{R}^n}(\Delta u)(\nabla_i\Delta^2 u)(\nabla_i\eta)\\
&\quad = -4\int_{\mathbb{R}^n}(\Delta^2 u)(\nabla_i\Delta u)(\nabla_i\eta)\;-2\int_{\mathbb{R}^n}(\Delta u)(\Delta^2 u)(\Delta\eta)\,.
\end{aligned}\tag{2.12}$$

We claim that

$$-4\int_{\mathbb{R}^n}(\Delta^2 u)(\nabla_i\Delta u)(\nabla_i\eta) \le \varepsilon E^2_\eta(u) + \frac{c(\varepsilon,n,s)}{\rho^4}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4}\,,\tag{2.13}$$

and

$$-2\int_{\mathbb{R}^n}(\Delta u)(\Delta^2 u)(\Delta\eta) \le \varepsilon E^2_\eta(u) + \frac{c(\varepsilon,n,s)}{\rho^4}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4}\,,\tag{2.14}$$

hold. Given the above estimates, we may conclude the required statement for the case $k = 1$ as follows. Choosing $\varepsilon = \frac{1}{2}$ in each of (2.13), (2.14) and combining with (2.12) we find

$$\frac{d}{dt}E^1_\eta(u) + 2E^2_\eta(u) \le E^2_\eta(u) + \frac{c(n,s)}{\rho^4}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4}\,,$$

whereupon subtraction of $E^2_\eta(u)$ from both sides yields the desired estimate.

The estimate (2.14) is (2.9) with $\delta_1 = \varepsilon/2$ and $k = 1$. It remains to prove the estimate (2.13). We compute

$$-4\int_{\mathbb{R}^n}(\Delta^2 u)(\nabla_i\Delta u)(\nabla_i\eta) \le \delta_1 E^2_\eta(u) + \frac{4c_\gamma^2 s^2}{\delta_1\rho^2}\int_{\mathbb{R}^n}|\nabla\Delta u|^2\gamma^{s-2}\,.\tag{2.15}$$

Now estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla\Delta u|^2\gamma^{s-2}\\
&\quad = -\int_{\mathbb{R}^n}(\Delta u)(\Delta^2 u)\gamma^{s-2} - (s-2)\int_{\mathbb{R}^n}(\Delta u)(\nabla_i\Delta u)(\nabla_i\gamma)\gamma^{s-3}\\
&\quad \le \rho^2\delta_2\int_{\mathbb{R}^n}|\Delta^2 u|^2\gamma^s + \frac{1}{2}\int_{\mathbb{R}^n}|\nabla\Delta u|^2\gamma^{s-2}\\
&\qquad\quad \frac{1}{2\rho^2}\Big(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\Big)\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4}\,.
\end{aligned}$$

Absorbing the second term on the right into the left we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n}|\nabla\Delta u|^2\gamma^{s-2}\\
&\quad \le 2\rho^2\delta_2\int_{\mathbb{R}^n}|\Delta^2 u|^2\gamma^s + \frac{1}{\rho^2}\Big(\frac{1}{2\delta_2} + c_\gamma^2(s-2)^2\Big)\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4}\,.
\end{aligned}\tag{2.16}$$

Combining (2.15) with (2.16) we find

$$
\begin{aligned}
&-4\int_{\mathbb{R}^n}(\Delta^2 u)(\nabla_i\Delta u)(\nabla_i\eta)\\
&\quad\le\Big(\delta_1+2\rho^2\delta_2\frac{4c_\gamma^2 s^2}{\delta_1\rho^2}\Big)E_\eta^2(u)+\frac{4c_\gamma^2 s^2}{\delta_1\rho^2}\bigg(\frac{1}{\rho^2}\Big(\frac{1}{2\delta_2}+c_\gamma^2(s-2)^2\Big)\bigg)\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4}\,.
\end{aligned}
$$

Choosing $\delta_1>0$ and $\delta_2>0$ such that $\Big(\delta_1+2\delta_2\frac{4c_\gamma^2 s^2}{\delta_1}\Big)\le\varepsilon$ yields (2.13).

Let us now continue by considering the case where $k\ge 2$. In this case we have $k-1\in\mathbb{N}$, and $s>4k$ implies $s-4>4k-4=4(k-1)$. Using these facts and Corollary 7.3, we see that

$$
\frac{1}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\le\frac{1}{\rho^4}\int_{\mathbb{R}^n}|\Delta^k u|^2\gamma^{s-4}+\frac{c(s,n)}{\rho^{4k}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}\,. \tag{2.17}
$$

and, using Corollary 7.3 again,

$$
\frac{1}{\rho^4}\int_{\mathbb{R}^n}|\Delta^k u|^2\gamma^{s-4}\le\delta_1\int_{\mathbb{R}^n}|\Delta^{k+1}u|^2\gamma^{s}+\frac{c(\delta_1,s,n)}{\rho^{4k}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}\,. \tag{2.18}
$$

Combining (2.17) with (2.18) and then choosing $\delta_1=\frac{1}{2c_3}$, where $c_3(n,s)$ is as in the previous Corollary, yields

$$
\frac{c_3}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\le\frac{1}{2}\int_{\mathbb{R}^n}|\Delta^{k+1}u|^2\gamma^{s-4}+\frac{\tilde c}{\rho^{4k}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}\,, \tag{2.19}
$$

for some $\tilde c=\tilde c(n,s)$. Using (2.19) to estimate the right hand side of Corollary 2.4 we find

$$
\begin{aligned}
\frac{d}{dt}E_\eta^k(u)+\frac{3}{2}E_\eta^{k+1}(u)&\le\frac{c_3}{\rho^8}\int_{\mathbb{R}^n}|\Delta^{k-1}u|^2\gamma^{s-8}\\
&\le\frac{1}{2}\int_{\mathbb{R}^n}|\Delta^{k+1}u|^2\gamma^{s-4}+\frac{\tilde c}{\rho^{4k}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}\,,
\end{aligned}
$$

which, after absorbing the first term on the right into the left, becomes

$$
\frac{d}{dt}E_\eta^k(u)+E_\eta^{k+1}(u)\le\frac{c_4}{\rho^{4k}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}
$$

as required.

□

3. A Blowup argument

Let $u:\mathbb{R}^n\times[0,T]\to\mathbb{R}$ be a smooth solution to (1.1). We consider the scale invariant quantity $e(u):B_1(0)\times[0,T]\to\mathbb{R}$ defined by

$$
e(u)(x,t):=d^4(x)|\Delta u|^2(x,t),
$$

where $d(x):=(1-|x|)$ is the distance from the boundary $\partial B_1(0)=\{x\in\mathbb{R}^n\,|\,|x|=1\}$ to the point x in $B_1(0)$. Note that $e=0$ on $\partial B_1(0)$. The function e is scale invariant in the following sense: If we define $\tilde u(\tilde x,\tilde t):=u(c\tilde x,c^4\tilde t)-c_0$, where c_0 is an arbitrary constant in $\mathbb{R}$, then $\tilde u:\mathbb{R}^n\times[0,\tilde T]\to\mathbb{R}$ is still a smooth solution to (1.1) and the quantity $\tilde e(\tilde x,\tilde t):=\tilde e(\tilde u)(\tilde x,\tilde t)$ which is defined by

$$
\tilde e(\tilde u)(\tilde x,\tilde t):=\tilde d^4(\tilde x)|\Delta\tilde u|^2(\tilde x,\tilde t),
$$

satisfies

$$\tilde{e}(\tilde{x},\tilde{t}) := e(x,t),$$

where here $x := c\tilde{x}$, $t := c^4\tilde{t}$, $\tilde{T} = \frac{T}{c^4}$, and $\tilde{d}(\tilde{x}) := (\frac{1}{c} - |\tilde{x}|)$ is the distance from $x \in B_{1/c}(0)$ to $\partial B_{1/c}(0)$. The scale invariance of e can be seen as follows:

$$(\nabla\tilde{u})(\tilde{x},\tilde{t}) = c(\nabla u)(x,t)\,, \qquad \text{and hence } (\nabla^k\tilde{u})(\tilde{x},\tilde{t}) = c^k(\nabla^k u)(x,t)\,,$$

$$\left(\frac{\partial}{\partial t}\tilde{u}\right)(\tilde{x},\tilde{t}) = c^4\left(\frac{\partial}{\partial t}u\right)(x,t)\,, \text{ and hence } \left(\left(\frac{\partial}{\partial t}\right)^k\tilde{u}\right)(\tilde{x},\tilde{t}) = c^{4k}\left(\left(\frac{\partial}{\partial t}\right)^k u\right)(x,t)\,,$$

$$\begin{aligned}\tilde{d}(\tilde{x}) &= \left(\frac{1}{c} - |\tilde{x}|\right)\\ &= \frac{1}{c}(1 - |c\tilde{x}|)\\ &= \frac{1}{c}(1-|x|) = \frac{1}{c}d(x)\,, \quad \text{and hence } \tilde{d}^4(\tilde{x}) = \frac{1}{c^4}d^4(x)\,,\end{aligned}$$

where here $(\frac{\partial}{\partial t})^k$ refers to k time derivatives, and $\nabla^k u$ refers to k spatial derivatives, and we are assuming that $k \in \mathbb{N}$ ($k \neq 0$). Therefore

$$|\Delta\tilde{u}|^2(\tilde{x},\tilde{t})\tilde{d}^4(\tilde{x}) = |c^2\Delta u|^2(x,t)\frac{1}{c^4}d^4(x) = |\Delta u|^2(x,t)d^4(x).$$

and $e(x,t) = \tilde{e}(\tilde{x},\tilde{t})$ as claimed. Note that

$$x \in B_1(0)\,,\ d^4(x) \geq Nt \quad \Longleftrightarrow \quad \tilde{x} \in B_{1/c}(0)\,,\ \tilde{d}^4(\tilde{x}) \geq N\tilde{t}$$

in view of the definitions of the terms involved.

In the following, we will assume that

(A1) $$b(u)(x,t) := t|\Delta u|^2(x,t) \leq k_0 < \infty \quad \text{for all } x \in \mathbb{R}^n\,, t \in [0,T],$$

for some fixed $k_0 \in \mathbb{R}^+$. That is, the quantity

$$Q(x,t) := Q(u)(x,t) := |\Delta u|^2(x,t) \tag{3.1}$$

may approach infinity as $t \searrow 0$, but it is only allowed to do so at a controlled, but non-integrable rate. Note that we have $e(x,t) = d^4(x)Q(x,t)$ and $b(x,t) = tQ(x,t)$. The function $b(x,t)$ is also scale invariant in the following sense: If we define $\tilde{u}$, $\tilde{x}$ and $\tilde{t}$ as above, then $\tilde{b}(\tilde{x},\tilde{t}) := b(\tilde{u})(\tilde{x},\tilde{t}) = b(x,t)$ and hence $\tilde{b}(\tilde{x},\tilde{t}) \leq k_0 < \infty$ for all $\tilde{x} \in \mathbb{R}^n$ and all $\tilde{t} \in [0,\tilde{T}]$. The scale invariance of b may be verified with an argument similar to the one we used above to show that e is scale invariant. We are interested in the local behaviour of solutions u to (1.1) which satisfy (A1). In particular, if at time zero $u_0 = u(\cdot,0)$ satisfies

(A2) $$\sup_{B_1(0)}\sum_{i=0}^{2n+2}|\nabla^i u_0|^2 \leq k_1 < \infty,$$

for some fixed $k_1 \in \mathbb{R}^+$, then we show that the solution satisfies estimates on a smaller ball for a short well-defined time interval. The following theorem is Theorem 1.1 of the introduction.

Theorem 3.1. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $T < \infty$ be a smooth solution to* (1.1) *which satisfies assumptions* (A1) *and* (A2). *Then there exists an $N = N(n,k_0,k_1) > 0$ such that*

$$e(x,t) \leq N$$

for all x, t which satisfy $x \in \overline{B}_1(0)$, $d^4(x) \geq Nt$, and $t \leq \frac{1}{N}$, $t \leq T$.

For our theorem on higher order regularity, we modify the quantities above. Let $s \in \mathbb{N}$, $s \geq 2$ be given and fixed, and define

$$\begin{aligned} Q_s(u)(x,t) &= (|\nabla^s u| + |\nabla^{s-1}u|^{s/(s-1)} + \ldots + |\nabla u|^s)^{4/s}(x,t) \\ e_s(u)(x,t) &= d^4(x)Q_s(u)(x,t) \\ b_s(u)(x,t) &= tQ_s(u)(x,t). \end{aligned}$$

These quantities are scale invariant in the sense explained above: for $\tilde{u}$, $\tilde{T}$, $\tilde{t}$, $\tilde{x}$ and $\tilde{d}$ defined as above, and $\tilde{Q}_s(\tilde{x},\tilde{t}) := Q_s(\tilde{u})(\tilde{x},\tilde{t})$ we have

$$\begin{aligned} \tilde{e}_s(\tilde{x},\tilde{t}) &:= \tilde{d}^4(\tilde{x})\tilde{Q}_s(\tilde{x},\tilde{t}) = d^4(x)Q_s(u)(x,t)\,, \quad \text{and} \\ \tilde{b}_s(\tilde{x},\tilde{t}) &:= \tilde{t}\tilde{Q}_s(\tilde{x},\tilde{t}) = tQ_s(x,t). \end{aligned}$$

For this set-up we require

$$b_s(u)(x,t) \leq k_0 < \infty \tag{$\mathrm{A_s1}$}$$

for all $x \in \mathbb{R}^n$ $t \in [0,T]$, and for some fixed $k_0 \in \mathbb{R}^+$, $s \geq 2$, $s \in \mathbb{N}$; and

$$\sup_{B_1(0)} \sum_{i=0}^{2n+s+1} |\nabla^i u_0|^2 \leq k_1 < \infty, \tag{$\mathrm{A_s2}$}$$

for some fixed $k_1 \in \mathbb{R}^+$. In this context we obtain the following variant of Theorem 3.1 above.

Theorem 3.2. *Let $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $T < \infty$ be a smooth solution to* (1.1) *which satisfies assumptions (A_s1) for some $s \in N, s \geq 2$ and (A_s2). Then there exists an $N = N(n, k_0, k_1, s) > 0$ such that*

$$e_s(x,t) \leq N\,, \tag{3.2}$$

for all x, t which satisfy $x \in \overline{B}_1(0)$, $d^4(x) \geq Nt$, $t \leq \frac{1}{N}$, $t \leq T$.

Note that this theorem is equivalent to Theorem 1.2 of the introduction. Under the same assumptions as Theorem 3.2, but with the condition that $s \geq 4$, we also obtain a local supremum bound for $|u|$:

Corollary 3.3. *Assume everything is as in Theorem 3.2 but that $s \geq 4$. Then we also have*

$$|u(x,t)| \leq \sqrt{k_1} + 1 \tag{3.3}$$

for all x, t which satisfy $x \in \overline{B}_1(0)$, $d^4(x) \geq Nt$, $t \leq \frac{1}{N}$, where N is as in the conclusion of Theorem 3.2 above.

Proof of Corollary 3.3. Let (x, t_0) be a point which satisfies $d^4(x) \geq Nt_0$ and $t_0 \leq \frac{1}{N}$. Then $d^4(x) \geq Nt$ and $t \leq \frac{1}{N}$ for all $t \leq t_0$. Hence, taking $s = 4$ in Theorem 3.2 we see that $|\Delta^2 u|(x,t)| \leq \frac{N}{d^4(x)}$ for all $t \leq t_0$ and that $|u(x,0)| \leq \sqrt{k_1}$ in view of ($\mathrm{A_s2}$). The evolution equation for $u(x,t)$ is $\frac{\partial}{\partial t}u(x,t) = -\Delta^2 u(x,t)$. Integrating this from 0 to t_0 and using the two estimates which we just derived, we see that $|u(x,t_0)| \leq t_0 \frac{N}{d^4(x)} + \sqrt{k_1}$. Using that $d^4(x) \geq Nt_0$ we obtain the result. $\square$

Now we prove Theorem 3.1.

Proof of Theorem 3.1. Define d resp. e to be zero on $\mathbb{R}^n \backslash B_1(0)$ resp. $(\mathbb{R}^n \backslash B_1(0)) \times [0,T]$. Then e is continuous on $\mathbb{R}^n \times [0,T]$. Assume that the conclusion of the theorem is false, and let $N \in \mathbb{N}$. Note that $e(x,0) \leq k_0$ where k_0 is the constant appearing in (A2). Without loss of generality $N > k_0$. The set of $x \in \overline{B_1(0)}$, $t \in [0,T]$ for which $1 \geq d^4(x) \geq Nt$ and $t \leq \frac{1}{N}$ is a compact set in $\mathbb{R}^n \times [0,T]$ which we denote by K. By compactness of K and continuity of e and the fact that $e(x,0) \leq k_0 < N$ for all $x \in \overline{B_1(0)}$, there must be a first time $t_0 \in (0, \frac{1}{N}]$ and (at least) one point $x_0 \in B_1(0)$ such that $e(x_0,t_0) = N$. That is: $e(x,t) < N$ for all $(x,t) \in K$ with $t < t_0$, and $e(x_0,t_0) = N$ for some point $(x_0,t_0) \in K$. Clearly we have $d(x_0) > 0$ for such a point, that is, $x_0 \in B_1(0)$, since $e(x_0,t_0) > 0$. Rescale the solution u to $\tilde{u}(\tilde{x},\tilde{t}) := u(c\tilde{x}, c^4\tilde{t}) - c_0$, where $c_0 := u(c\tilde{x}_0, 0)$, and $c > 0$ is chosen so that $\tilde{d}^4(\tilde{x}_0) = N$. It is possible to choose c in this way: $\tilde{d}(\tilde{x}) = \frac{1}{c}d(x)$, so we choose $c^4 = \frac{d^4(x_0)}{N}$, which is larger than zero since $d(x_0) > 0$ as we explained above. Our choice of c_0 guarantees that $\tilde{u}(\tilde{x}_0, 0) = 0$. Note for later use that $c^4 = \frac{d^4(x_0)}{N} \leq \frac{1}{N}$ (≤ 1), and $c \searrow 0$ as $N \to \infty$. Now

$$N = e(x_0,t_0) = \tilde{e}(\tilde{x}_0,\tilde{t}_0) = \tilde{d}^4(\tilde{x}_0)\tilde{Q}(\tilde{x}_0,\tilde{t}_0) = N\tilde{Q}(\tilde{x}_0,\tilde{t}_0)$$

due to scaling, and hence

$$\tilde{Q}(\tilde{x}_0,\tilde{t}_0) = 1.$$

Similarly,

$$N \geq e(x,t) = \tilde{e}(\tilde{x},\tilde{t}) = \tilde{d}^4(\tilde{x})\tilde{Q}(\tilde{x},\tilde{t})$$

for all $(x,t) \in K$ with $t \leq t_0$, implies

$$\tilde{Q}(\tilde{x},\tilde{t}) \leq \frac{N}{\tilde{d}^4(\tilde{x})} \tag{3.4}$$

for all $\tilde{x} \in B_{1/c}(0)$ with $\tilde{d}^4(\tilde{x}) \geq \tilde{t}N$ and $\tilde{t} \leq \tilde{t}_0$. Note that the inequality (3.4) is also valid for all $\tilde{x}$ with $\tilde{d}^4(\tilde{x}) \geq \tilde{t}N$ and $\tilde{t} \leq \tilde{t}_0$, since $\tilde{d}(\tilde{x}) = 0$ outside of $B_{1/c}(0)$ (here we define $\frac{M}{0} = \infty$ for $M > 0$). As in the paper [Si] we consider two cases: **Case 1**, where $\tilde{d}^4(\tilde{x}_0) \geq 2N\tilde{t}_0$ (which is equivalent to $\tilde{t}_0 \leq \frac{1}{2}$, since $\tilde{d}^4(\tilde{x}_0) = N$), and **Case 2**, where $\tilde{d}^4(\tilde{x}_0) < 2N\tilde{t}_0$ (which is equivalent to $1 \geq \tilde{t}_0 > \frac{1}{2}$, since $\tilde{t}_0 N \leq \tilde{d}^4(\tilde{x}_0) < 2N\tilde{t}_0$ and $\tilde{d}^4(\tilde{x}_0) = N$. Note that $N\tilde{t}_0 \leq \tilde{d}^4(\tilde{x}_0)$ since $(x_0,t_0) \in K$). We start with Case 1.

Case 1: In this case we have $\tilde{d}^4(\tilde{x}_0) \geq 2N\tilde{t}_0$. For $\tilde{y}$ with $\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}$, we obtain

$$\tilde{d}^4(\tilde{y}) \geq \frac{N}{2} \geq N\tilde{t}_0 \geq N\tilde{t} \tag{3.5}$$

for all $\tilde{t} \leq \tilde{t}_0$. Hence, we see that

$$\tilde{Q}(\tilde{y},\tilde{t}) \leq \frac{N}{\tilde{d}^4(\tilde{y})} \leq 2 \tag{3.6}$$

for all $\tilde{t} \leq \tilde{t}_0$ in view of (3.5) and (3.4).

We also have that $\tilde{d}^4(\tilde{x}_0) = N \geq \frac{N}{2}$ and so the above estimate also holds for $\tilde{y} = \tilde{x}_0$ and $\tilde{t} = \tilde{t}_0$. We calculate

$$\begin{aligned}
& & \frac{N}{2} \leq \tilde{d}^4(\tilde{y}) = \left(\frac{1}{c} - |\tilde{y}|\right)^4 \\
&\iff & \left(\frac{1}{c} - |\tilde{y}|\right) \geq \frac{N^{\frac{1}{4}}}{2^{\frac{1}{4}}} \\
&\iff & |\tilde{y}| \leq -\frac{N^{\frac{1}{4}}}{2^{\frac{1}{4}}} + \frac{1}{c}\,. \qquad (3.7)
\end{aligned}$$

Furthermore $\tilde{d}^4(\tilde{x}_0) = N$ implies $|\tilde{x}_0| = -N^{\frac{1}{4}} + \frac{1}{c}$. Assume that $\tilde{y}$ is an arbitrary point with $\tilde{y} \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$. Then we have

$$\begin{aligned}
|\tilde{y}| \leq |\tilde{x}_0| + |\tilde{x}_0 - \tilde{y}| &= -N^{\frac{1}{4}} + \frac{1}{c} + |\tilde{x}_0 - \tilde{y}| \\
&\leq -N^{\frac{1}{4}} + \frac{1}{c} + \frac{N^{\frac{1}{4}}}{400} \\
&\leq \frac{1}{c} - \frac{N^{\frac{1}{4}}}{2^{\frac{1}{4}}} \qquad (3.8)
\end{aligned}$$

and hence, in view of (3.7)

$$\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}\,. \qquad (3.9)$$

Therefore $\tilde{y} \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$ and $\tilde{t} \leq \tilde{t}_0 \leq \frac{1}{2}$ implies in Case 1 that

$$\tilde{Q}(\tilde{y}, \tilde{t}) \leq 2\,, \quad \text{and} \quad \tilde{Q}(\tilde{x}_0, \tilde{t}_0) = 1\,,$$

in view of (3.6) and the definition of $\tilde{x}_0$ and $\tilde{t}_0$.

Case 2. In this case we have $1 \geq \tilde{t}_0 > \frac{1}{2}$. For all $\tilde{t} \leq \frac{1}{2}$ and $\tilde{y}$ with $\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}$ we have

$$\tilde{d}^4(\tilde{y}) \geq \frac{N}{2} \geq N\tilde{t}$$

and hence

$$\tilde{Q}(\tilde{y}, \tilde{t}) \leq \frac{N}{\tilde{d}^4(\tilde{y})} \leq 2 \qquad (3.10)$$

in view of (3.4). For $\tilde{t}_0 \geq \tilde{t} \geq \frac{1}{2}$ we have

$$\tilde{Q}(\tilde{y}, \tilde{t}) \leq \frac{k_0}{\tilde{t}} \leq 2k_0, \qquad (3.11)$$

in view of (A1). Note that we may assume without loss of generality that $k_0 \geq 1$. Now we know from (3.8) that $y \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$ implies that $\tilde{d}^4(\tilde{y}) \geq \frac{N}{2}$. Hence, using the inequalities (3.10) and (3.11), we see that

$$\tilde{Q}(\tilde{y}, \tilde{t}) \leq 2k_0\,, \quad \text{and} \quad \tilde{Q}(\tilde{x}_0, \tilde{t}_0) = 1\,, \qquad (3.12)$$

for all $y \in B_{N^{\frac{1}{4}}/400}(\tilde{x}_0)$ and $t \in [0, \tilde{t}_0]$. We have shown that in both Case 1 and Case 2 we obtain the estimate (3.12). Now we use Corollary 2.5 to obtain a contradiction.

We use $v : B_R(0) \times [0, \tilde t_0] \to \mathbb{R}$ to denote the rescaled solution $\tilde u : B_{N^{\frac14}/400}(\tilde x_0) \times [0, \tilde t_0] \to \mathbb{R}$. That is $v(\cdot,\cdot) = \tilde u(\cdot - \tilde x_0, \cdot)$, $R = N^{\frac14}/400$, $\tilde t_0 \le 1$. The definition of $\tilde u$ guarantees that $\tilde u(\tilde x_0, 0) = 0$, and hence we have $v(0,0) = 0$. Also, using (A2) and the fact that $c^4 \le 1/N$ and $N > 1$, we see that

$$
\begin{aligned}
\sup_{B_{1/c}(0)} \sum_{i=1}^{2n+2} |\nabla^i v|^2(\cdot,0) &= \sup_{B_1(x_0)} \sum_{i=1}^{2n+2} c^{2i} |\nabla^i u|^2(\cdot,0) \\
&\le \sup_{B_1(x_0)} \sum_{i=1}^{2n+2} \frac{1}{N^{i/2}} |\nabla^i u(\cdot,0)|^2 \\
&\le \frac{k_1}{N^{1/2}}.
\end{aligned} \tag{3.13}
$$

Hence, combining this estimate with the fact that $v(0,0) = 0$ and by choosing N sufficiently large, we may assume w.l.o.g. that

$$
\sup_{B_1(0)} \sum_{i=0}^{2n+2} |\nabla^i v|^2(\tilde x, 0) \le \tilde\varepsilon(N) \tag{3.14}
$$

where $\tilde\varepsilon(N) \to 0$ as $N \to \infty$ (k_0, k_1, n are fixed in this argument). Define $2\rho = R = N^{\frac14}/400 \le 1/c$ ($c \le \frac{1}{N^{1/4}}$ as we mentioned above) and $p = p(n) = n+1$. Then $\rho \to \infty$ as $N \to \infty$. Corollary 2.5 implies that

$$
\frac{d}{dt} E^p_\eta(v) + E^{p+1}_\eta(v) \le \frac{C}{\rho^{4p}} \int_{\mathbb{R}^n} |\Delta v|^2 \gamma^{s-4p} \tag{3.15}
$$

for all $t \le \tilde t_0$ for all $s > 4p + 4$, where $\eta = \gamma^s$, and γ is a cutoff function as in (γ), and $C = C(n,s)$. Choose $s = 4p(n) + 5 = 4n + 9$, so that $C = C(n)$. We know from the estimate (3.12) that $Q(v) = |\Delta v|^2 \le 2k_0$ on $B_R(0) \times [0, \tilde t_0]$ and hence, combining this with (3.15) we have

$$
\begin{aligned}
\frac{d}{dt} E^p_\eta(v) + E^{p+1}_\eta(v) &\le \frac{C}{\rho^{4p}} \int_{\mathbb{R}^n} |\Delta v|^2 \gamma^{s-4p} \\
&\le \frac{C}{\rho^{4p}} \int_{B_{2\rho}} 2k_0 \\
&= 2\omega_n k_0 C \rho^{n-4p}
\end{aligned} \tag{3.16}
$$

which implies that

$$
E^p_\eta(v)(t) \le (2\omega_n k_0 C + \omega_n k_1)\rho^{n-4p} = (2\omega_n k_0 C + \omega_n k_1)\rho^{-3n-4}
$$

for all $t \leq \tilde{t}_0 \leq 1$ in view of the fact that

$$\begin{aligned} E^p_\eta(v)(0) = \int_{\mathbb{R}^n} |\Delta^p v_0|^2(\tilde{x})\gamma^s d\tilde{x} &\leq \int_{B_{2\rho}(0)} |\Delta^p v_0|^2(\tilde{x}) d\tilde{x} \\ &\leq \int_{B_{1/c}(0)} |\Delta^p v_0|^2(\tilde{x}) d\tilde{x} \\ &= \int_{B_{1/c}(0)} c^{4p}|\Delta^p u_0|^2(c\tilde{x}) d\tilde{x} \\ &= c^{4p-n}\int_{B_1 0} |\Delta^p u_0|^2(x)dx \\ &\leq k_1\omega_n c^{4p-n} \\ &\leq k_1\omega_n \rho^{n-4p} \end{aligned}$$

where we have used assumption (A2) again, the definition of v, the scaling properties of the derivatives of $u(cx, 0)$, and the fact that $1/c \geq 2\rho \geq \rho$. In particular,

$$\int_{B_1(0)} |\Delta^p v|^2 \leq (\omega_n 2k_0 C + k_1\omega_n)\rho^{-3n-4} \to 0$$

as $\rho \to \infty$, in view of the fact that $p = p(n) = n + 1$. We have shown that

$$\int_{B_1(0)} |\Delta^p v|^2 \leq \varepsilon_p(k_0, k_1, n, \rho)$$

for all $t \leq \tilde{t}_0 \leq 1$ where $\varepsilon_p(k_0, k_1, n, \rho) \to 0$ as $\rho \to \infty$, that is, as $N \to \infty$. We can similarly show that

$$\int_{B_1(0)} |\Delta^{p-1} v|^2 \leq \varepsilon_{p-1}(k_0, k_1, n, \rho).$$

We also have

$$\begin{aligned} \frac{d}{dt}\int_{B_1(0)} |\Delta^{p-2} v|^2 &= 2\int_{B_1(0)} (\Delta^{p-2} v)(\Delta^p v) \\ &\leq \int_{B_1(0)} |\Delta^{p-2} v|^2 + \int_{B_1(0)} |\Delta^p v|^2 \\ &\leq \int_{B_1(0)} |\Delta^{p-2} v|^2 + \varepsilon_p \end{aligned}$$

in view of Young's inequality, and the estimate just shown, and hence, after integrating in time from 0 to $\tilde{t}_0 \leq 1$ we see that

$$\int_{B_1(0)} |\Delta^{p-2} v|^2 \leq \varepsilon_{p-2}(N) \tag{3.17}$$

for all $t \in [0, \tilde{t}_0 \leq 1]$ with $\varepsilon_{p-2}(N) \to 0$ as $N \to \infty$: we leave out dependence on k_1, k_0, n since these variables are fixed. More explictly: $f(t) := e^{-t}(\int_{B_1(0)} |\Delta^{p-2} v|^2)(t) - 2t\varepsilon_p$ satisfies $\frac{df}{dt}(t) \leq 0$ for all $0 \leq t \leq \tilde{t}_0$ and $f(0) = (\int_{B_1(0)} |\Delta^{p-2} v|^2)(0) \leq \tilde{\varepsilon}(N) \to 0$ as $N \to \infty$ in view of (3.14), and so, integrating f from 0 to t_0, we see that the estimate (3.17) is true.

Continuing in this way, we get, for N sufficiently large:

$$\int_{B_1(0)} |\Delta^l v|^2 \leq \varepsilon_l(N) \tag{3.18}$$

for $l = p, p-2, p-4, \ldots, 1$ (or 0). Starting with $p-1$ instead of p, we similarly get

$$\int_{B_1(0)} |\Delta^l v|^2 \leq \varepsilon_l(N) \tag{3.19}$$

for $l = p-1, p-3, p-5, \ldots, 1$ (or 0), where $\varepsilon_l(N) \to 0$ as $N \to \infty$. That is

$$\int_{B_1(0)} |\Delta^l v|^2 \leq \varepsilon_l(N) \tag{3.20}$$

for $l \in \{0, \ldots, p\}$, where $\varepsilon_l(N) \to 0$ as $N \to \infty$.

Using the L^2 estimates, Lemma 7.1 from the Appendix, and the estimate (3.20) we see that

$$\int_{B_{1/2}(0)} |\nabla^l v|^2 \leq \hat{\varepsilon}(N) \tag{3.21}$$

for all $0 \leq l \leq 2p = 2n+2$, where $\hat{\varepsilon}(N) \to 0$ as $N \to \infty$ (choose $\sigma = \sigma(n) = \frac{1}{4p(n)}$, so that $1 - 2p\sigma = 1/2$).

Applying the Sobolev-Morrey inequality [E, Theorem 6, Section 5.6.3], with p, k there equal to 2, and $2n+2$ respectively, we see that

$$\tilde{Q}(0, t_0) = |\Delta v|^2(0, t_0) \leq C(n) \Big(\sum_{l=0}^{2n+2} \int_{B_{1/2}(0)} |\nabla^l v|^2(\cdot, t_0) \Big)^{\frac{1}{2}} \leq C(2n+3)^{\frac{1}{2}} (\hat{\varepsilon})^{\frac{1}{2}}$$

and hence

$$\tilde{Q}(0, t_0) \to 0$$

as $N \to \infty$. This contradicts the fact that $\tilde{Q}(0, t_0) = 1$.

□

The proof of Theorem 3.2 is essentially the same as the proof of Theorem 3.1.

Proof of Theorem 3.2. Replace $Q(x,t)$ by $Q_s(x,t)$, $e(x,t)$ by $e_s(x,t)$, $b(x,t)$ by $b_s(x,t)$, and $Q(u)$ by $Q_s(u)$ and repeat the above proof. At the point where $|\Delta v|^2 = Q(v) \leq 2k_0$ on $B_R(0) \times [0, \tilde{t}_0]$ is used in the inequality (3.16), use instead the fact that $|\Delta v|^2 \leq |\nabla^2 v|^2 \leq Q_s(v) \leq 2k_0$. Also choose $p(n) = n + (s/2)$ or $p = n + \frac{(s+1)}{2}$ in the proof: whichever is an integer. The last part of the proof, where Morrey's embedding Theorem is used, has to be slightly modified: $Q_s(v)(0, t_0) = 1$ implies that $|\nabla^r v|(0, t_0) \geq \delta(s) > 0$ for some $r \in \{1, \ldots, s\}$ for some small $\delta(s) > 0$: otherwise the sum of the terms appearing in $Q_s(v)(0, t_0)$ would be less than one.

Applying the Sobolev-Morrey inequality [E, Theorem 6, Section 5.6.3] with p, k there equal to 2, $2n+s$ respectively, we see that

$$\begin{aligned} 0 < \delta^2(s) &\leq |\nabla^r v|^2(0, t_0) \\ &\leq C(n,s) \Big(\sum_{l=0}^{2n+s} \int_{B_{1/2}(0)} |\nabla^l v|^2(\cdot, t_0) \Big)^{1/2} \\ &\leq C(2n+1+s)^{\frac{1}{2}} (\hat{\varepsilon}(N))^{1/2}, \end{aligned}$$

which leads to a contradiction if N is chosen large enough, since $\hat{\varepsilon}(N) \to 0$ as $N \to \infty$. □

4. Uniqueness

In this section we prove that smooth solutions to (1.1) which satisfy $|\Delta u|^2(\cdot,t) \le \frac{k_0}{t}$ are uniquely determined by their initial values.

Theorem 4.1. *Let $v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $T < \infty$ be a smooth solution to* (1.1) *which satisfies*

$$|\Delta v|^2(x,t) \le \frac{k_0}{t} \tag{4.1}$$

for some $k_0 \in \mathbb{R}$, for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$ and

$$v_0 \equiv 0. \tag{4.2}$$

Then $v \equiv 0$.

Proof. Since

$$\sup_{B_1(0)} \sum_{i=0}^{p} |\nabla^i v_0|^2 = 0$$

for any $p > 0$, (c.f. (A2)), Theorem 3.1 tells us that $|\Delta v|^2(0,t) \le 2N(n,k_0)$ for some $N = N(n,k_0) \in \mathbb{R}$ for all $t \le 1/N$. By setting $\tilde{v}(\cdot,t) = v(\cdot - x_0, t)$ and using Theorem 3.1 for $\tilde{v}$, we see that $|\Delta v|^2(x_0,t) \le N(n,k_0)$ for all $t \le 1/N$, for all $x_0 \in \mathbb{R}^n$. Corollary 2.5 implies that

$$\frac{\partial}{\partial t} E^p_\eta(v) + E^{p+1}_\eta(v) \le 2CN\omega_n \rho^{n-4p} = 2CN\omega_n\rho^{-3n-4} \tag{4.3}$$

where p is now fixed and chosen to be $p(n) = n+1$, and η is a non-negative cutoff function with $\eta = 1$ on $B_\rho(0)$, and $C = C(n)$. To see this repeat the argument from the inequality (3.15) up to (3.16) but use this v (instead of the v appearing there) and use the fact that $k_0 = 0$, and $|\Delta v|^2 \le N$ for all $t \le 1/N$ for this v. This implies that $E^p_\eta(v)(t) \le 2CN\omega_n\rho^{-3n-4}$ for all $t \le 1/N \le 1$, since $E^p_\eta(v)$ is non-negative, and $E^p_\eta(v)(0) = 0$. Letting $\rho \to \infty$, we see that $\int_{\mathbb{R}^n} |\Delta^p v|^2 = 0$ for all $t \le 1/N$. Similarly $\int_{\mathbb{R}^n} |\Delta^{p-1} v|^2 = 0$ for all $t \le 1/N$. Now use

$$\begin{aligned} \frac{\partial}{\partial t} \int_{B_1(0)} |\Delta^{p-2} v|^2 &= 2\int_{B_1(0)} \Delta^p v \Delta^{p-2} v \\ &\le \int_{B_1(0)} |\Delta^p v|^2 + |\Delta^{p-2} v|^2 = \int_{B_1(0)} |\Delta^{p-2} v|^2 \end{aligned}$$

which tells us, after integrating, that $\int_{B_1(0)} |\Delta^{p-2} v|^2 = 0$ for all $t \le 1/N$. Differentiating $\int_{B_1(0)} |\Delta^{p-3} v|^2$ w.r.t. time and using $\int_{B_1(0)} |\Delta^{p-1} v|^2 = 0$ we obtain, using the same argument, that $\int_{B_1(0)} |\Delta^{p-3} v|^2 = 0$ for all $t \le 1/N$. Continuing in this way, we find that $\int_{B_1(0)} |\Delta^l v|^2 = 0$ for all $0 \le l \le p$, for all $t \le 1/N$. Similarly, we obtain $\int_{B_1(x_0)} |\Delta^l v|^2 = 0$ for all $0 \le l \le p$, for all $t \le 1/N$ for all $x_0 \in \mathbb{R}^n$. In particular, by choosing $l = 0$, we see that $v(\cdot,t) = 0$ for all $t \le 1/N$, $t \le T$. Repeating this argument for the function $\tilde{v}(\cdot,\tilde{t}) = v(\cdot,\tilde{t} + 1/N)$, we see that $v(\cdot,t) = 0$ for all $t \le T$, as required. □

Corollary 4.2. *Let $u, v : \mathbb{R}^n \times [0,T] \to \mathbb{R}$, $T < \infty$ be smooth solutions to* (1.1) *which satisfy*

$$|\Delta v|^2(x,t) + |\Delta u|^2(x,t) \le \frac{k_0}{t} \tag{4.4}$$

for all $t \in [0,T]$ and all $x \in \mathbb{R}^n$ and

$$u_0(\cdot) = v_0(\cdot). \tag{4.5}$$

Then $u \equiv v$.

Proof. Set $s = u - v$. Then s satisfies the conditions of Theorem 4.1. Hence $s \equiv 0$ as required. □

5. A Tychonoff-type solution and Non-uniqueness

In this section we describe a simple modification to the classical Tychonoff counterexample, see [T], which establishes non-uniquness for complete solutions of the polyharmonic heat equation. We follow the construction given in [J, Chapter 7, Section 1 (a), pp 211–213].

Let $k \in \mathbb{N}$ and consider a solution $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ to

$$(\partial_t - \Delta^k)u = 0 \qquad \text{on} \quad \mathbb{R}^n \times [0,T]\,, \tag{5.1}$$

$$u(\cdot,0) = u_0(\cdot) \qquad \text{on} \quad \mathbb{R}^n\,. \tag{5.2}$$

We shall construct infinitely many solutions to (5.1)-(5.2) which have zero as their initial data.

For functions $g_j : [0,T] \to \mathbb{R}$ to be chosen, set

$$u(x,t) = \sum_{j=0}^{\infty} g_j(t) x_1^{2jk}\,.$$

The convergence of this series will be guaranteed by our choice of g_j, and verified later. Differentiating formally, we find

$$\begin{aligned}\sum_{j=0}^{\infty} (\partial_t g_j)(t) x_1^{2jk} &= \partial_t u(x,t) = \Delta^k u(x,t) \\ &= \sum_{j=1}^{\infty} (2jk)(2jk-1)\cdots(2jk-2k+1) g_j(t) x_1^{2jk-2k} \\ &= \sum_{j=0}^{\infty} (2jk+2k)(2jk+2k-1)\cdots(2jk+1) g_{j+1}(t) x_1^{2jk}\end{aligned}$$

for all $j \in \mathbb{N}_0$. We are thus led to the recurrence relation

$$(\partial_t g_j) = (2jk+2k)(2jk+2k-1)\cdots(2jk+1) g_{j+1} \tag{5.3}$$

for all $j \in \mathbb{N}_0$. We set $g_j(t) = \lambda(j,k) g_0^{(j)}(t)$, where $(g_0)^j$ refers to j temporal derivatives of g_0, and $\lambda(j,k)$ is a constant to be determined depending only on j,k. Using this choice of g_j, we see that (5.3) is satisfied, provided that

$$g_0^{(j+1)} \lambda(j,k) = (2jk+2k)(2jk+2k-1)\cdots(2jk+1) \lambda(j+1,k) g_0^{(j+1)}\,,$$

which for $g_0^{(j+1)} \neq 0$ is equivalent to

$$\frac{\lambda(j,k)}{\lambda(j+1,k)} = (2jk+2k)(2jk+2k-1)\cdots(2jk+1)\,.$$

Using $\lambda(0,k) = 1$, we see that this implies that $\lambda(j,k) = \frac{1}{(2jk)!}$ for all $j \in \mathbb{N}_0$ (we use $0! := 1$) . Let us now set

$$g_0(t) = \exp(-t^{-p})$$

for $t > 0$ and $p > 1$.

Lemma 5.1. *There is an absolute constant $\varepsilon_0 > 0$ and a $p > 1$ such that the estimate*

$$\left|g_0^{(j)}(t)\right| \leq \frac{j!2^j}{t^j} \exp\big(- \varepsilon_0 (2t)^{-p}\big)$$

holds for all $t > 0$.

Proof. Consider the function $h(z) = \exp(-z^{-p})$ for $p > 1$. Since $z^p := \exp(p\mathrm{Log(z)})$ is analytic on $\mathbb{C}\backslash(-\infty, 0]$, h is analytic on $\mathbb{C}\backslash(-\infty, 0]$: for polar coordinates $z = re^{i\theta}$ with $\theta \in (-\pi, \pi)$ we are using $\mathrm{Log(z)} := \log(\mathrm{r}) + \mathrm{i}\theta$, and hence $z^p = r^p e^{ip\theta}$. For $0 < \rho < t$, Cauchy's integral formula on $S_\rho(t + 0i) = S_\rho(t)$, the circle in $\mathbb{C}$ with radius ρ centred at $t + 0i$, gives

$$g_0^{(j)}(t) = h^{(j)}(t + 0i) = \frac{j!}{2\pi i} \int_{S_\rho(t)} \frac{h(z)}{(z - t)^{j+1}} dz\,.$$

This gives the estimate

$$|g_0^{(j)}(t)| \leq \frac{j!}{\rho^j} \sup_{z\in S_\rho(t)} |h(z)|\,. \tag{5.4}$$

In polar coordinates $z = r\exp(i\theta)$, $\theta \in (-\pi, \pi)$ we have

$$h(z) = \exp\big(- r^{-p}e^{-ip\theta}\big) = \exp\big(- r^{-p}(\cos(p\theta) - i\sin(p\theta))\big)\,.$$

Therefore

$$|h(z)| \leq \exp(-r^{-p}\cos p\theta)\,. \tag{5.5}$$

For $z \in S_\rho(t)$, we have

$$-\frac{\pi}{2} < -\theta_0 \leq \theta \leq \tan^{-1}(\rho/\sqrt{t^2 - \rho^2}) =: \theta_0 < \frac{\pi}{2}$$

Note that θ_0 doesn't depend on p. So we may choose $p > 1$ such that $p\theta_0 < \frac{\pi}{2}$: this is possible since $\theta_0 < \frac{\pi}{2}$. We then have $\cos(p\theta) \geq \cos(p\theta_0) =: \varepsilon_0 > 0$ for all $\theta \in (-\theta_0, \theta_0)$. Since $r \leq 2t$ we may estimate

$$-r^{-p}\cos p\theta \leq -\varepsilon_0(2t)^{-p}$$

for all $\theta \in (-\theta_0, \theta_0)$, which combined with our earlier estimate (5.5) yields

$$\sup_{z\in S_\rho(t)} |h(z)| \leq \exp\big(- \varepsilon_0(2t)^{-p}\big)\,.$$

Inserting this into the estimate (5.4) and choosing $\rho = t/2$ finishes the proof. □

Lemma 5.1 implies

$$
\begin{aligned}
|u(x,t)| &\le \sum_{j=0}^{\infty} |g_j(t)||x_1|^{2jk} \\
&= \sum_{j=0}^{\infty} \frac{|g_0^{(j)}(t)|}{(2jk)!}|x_1|^{2jk} \\
&\le \sum_{j=0}^{\infty} \frac{j!2^j}{t^j(2jk)!}|x_1|^{2jk} \exp\big(-\varepsilon_0(2t)^{-p}\big) \\
&\le \exp\big(-\varepsilon_0(2t)^{-p}\big) \sum_{j=0}^{\infty} \frac{j!}{(2jk)!}\Big(\frac{|x_1|^{2k}}{t/2}\Big)^j \\
&\le \exp\big(-\varepsilon_0(2t)^{-p}\big) \sum_{j=0}^{\infty} \frac{1}{j!}\Big(\frac{|x_1|^{2k}}{t/2}\Big)^j \\
&= \exp\Big(-\frac{\varepsilon_0}{(2t)^p} + \frac{|x_1|^{2k}}{t/2}\Big).
\end{aligned}
$$

Here we have used $\frac{j!}{(2j)!} \le \frac{1}{j!}$ for all $j \in \mathbb{N}_0$ which may be seen using induction. Therefore u is well-defined for every $t > 0$. Moreover, $p > 1$ implies that the first term above always dominates for small t and so u converges uniformly to zero on compact subsets of $\mathbb{R}^n$ as $t \searrow 0$. More precisely, let K be a compact subset of $\mathbb{R}^n$ with diameter d and $0 \in K$. Then $|x| \le d$ and for $x \in K$

$$
\lim_{t\searrow 0} |u|(x,t) \le \lim_{t\searrow 0} \exp\Big(-\frac{\varepsilon_0}{(2t)^p} + \frac{d^{2k}}{t/2}\Big) = 0\,.
$$

A similar argument shows that all derivatives of u exist and converge uniformly to zero on compact subsets of $\mathbb{R}^n$ as $t \searrow 0$. We explain this in the following. Assuming $x = x_1$ satisfies $|x| \le d$ where $d \ge 1$ and taking s spatial derivatives formally we find

$$
\begin{aligned}
|(\partial_x)^s u(x,t)| &= \left| \sum_{j\ge s/(2k)} (g_0)^j(t)(x)^{2jk-s}\frac{(2jk)(2jk-1)\dots(2jk-s+1)}{(2jk)!}\right| \\
&\le \sum_{j\ge s/(2k)} \big|(g_0)^j(t)(x)^{2jk-s}\big| \left|\frac{(2jk)(2jk-1)\dots(2jk-s+1)}{(2jk)!}\right| \\
&\le \sum_{j\ge s/(2k)} |g_0^j|(t)|d|^{2jk-s}\frac{1}{(2jk-s)!} \\
&\le \sum_{j\ge s/(2k)} |g_0^j|(t)(|d|^{2k})^j\frac{1}{(2jk-s)!} \\
&\le \exp(-\varepsilon_0(2t)^{-p}) \sum_{j\ge s/(2k)} (2d^{2k}/t)^j\frac{j!}{(2jk-s)!} \\
&\le \exp(-\varepsilon_0(2t)^{-p}) \sum_{\{j\ \mid\ 2kj\ge s\}} (2d^{2k}/t)^j\frac{(kj)!}{(2jk-s)!}
\end{aligned}
$$

$$
\begin{aligned}
&\leq s! \exp(-\varepsilon_0(2t)^{-p}) \sum_{\{j \ | \ 2kj \geq s\}} \frac{(2d^{2k}/t)^j}{kj!} \\
&\leq s! \exp(-\varepsilon_0(2t)^{-p}) \sum_{\{j \ | \ 2kj \geq s\}} \frac{(2d^{2k}/t)^j}{j!} \\
&\leq s! \exp(-\varepsilon_0(2t)^{-p}) \exp(2d^{2k}/t),
\end{aligned}
$$

which goes to zero as $t \searrow 0$. Here we used that $\frac{(r!)^2}{(2r-s)!} \leq s!$ for all $r \geq s, r, s \in \mathbb{N}_0$, which may be verified using induction on r. Since s time derivatives of u are formally given by $2ks$ spatial derivatives of u, we see that all mixed derivatives (space and time) of u exist for $t > 0$ and converge uniformly on (spatial) compact sets $K \subset \mathbb{R}^n$ to 0.

By extending u to be zero for all $t \leq 0$ we have a solution $u \in C^\infty(\mathbb{R}^n \times (-\infty, \infty))$ to (5.1)-(5.2) which is non-zero for $t > 0$ and satisfies $u \equiv 0$ for all $t \leq 0$.

6. An example

Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be given by $u_0(x_1, x_2, \dots, x_n) := 1$ if $x_1 > 0$, $u_0(x_1, x_2, \dots, x_n) := 0$ if $x_1 \leq 0$. Setting $u(x,t) := \int_{\mathbb{R}^n} u_0(x-y) b(y,t) dy$, with $b : \mathbb{R}^n \times (0, \infty) \to \mathbb{R}$ the bi-harmonic heat kernel on $\mathbb{R}^n$, we see that the function $u : \mathbb{R}^n \times (0,T) \to \mathbb{R}$ is smooth and solves $\frac{\partial}{\partial t} u(x,t) = -\Delta^2 u(x,t)$ for all $t > 0$ for all $x \in \mathbb{R}^n$, and that $u(\cdot, t) \to u_0(\cdot)$ uniformly on any compact set K contained in $\mathbb{R}^n \backslash \{x \in \mathbb{R}^n | x_1 = 0\}$. Furthermore, there exists a $k_0 > 0$ such that for all $s > 0$ there exists a $x_s \in \mathbb{R}^n$ such that $|\Delta u|^2(x_s, s) = \frac{k_0}{s}$. The biharmonic heat kernel b is given by

$$
b(x,t) = (2\pi)^{-n/2} t^{-n/4} \int_{\mathbb{R}^n} e^{i\langle w, x \rangle t^{-1/4} - |w|^4} dw.
$$

We verify of all these facts below.

We have (see the Appendix of [KL], and the papers [GP], [GG1],[GG2] for further, related and similar estimates)

$$
\left| \left(\frac{\partial}{\partial t} \right)^l \left(\nabla^k \right) b(x,t) \right| \leq C(k,l,m) (t^{-p(k,l) + m/4} + t^{(m-1)/4}) |x|^{-m} \tag{6.1}
$$

for all $l, k \in \mathbb{N}_0, m \in \mathbb{N}_0$, for some $p(k,l) \in \mathbb{N}$ for all $x \in \mathbb{R}^n$ for all $t > 0$. This can be seen as follows. Let $f : \mathbb{R}^n \to \mathbb{R}$ be $f(y) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\langle w, y \rangle - |w|^4} dw$, so that $b(x,t) = t^{-n/4} f(-\frac{x}{t^{1/4}})$. Then f is the Fourier transform of the function $l : \mathbb{R}^n \to \mathbb{R}$, $l(w) := e^{-|w|^4}$ which is in $\mathcal{S}$, the so called Schwartz Space (see [SW] Chapter I.3 where this set of functions is defined and called the space of *testing functions*). Hence f itself is in $\mathcal{S}$ (see [SW], Theorem 3.2 Chapter I.3), in particular $|\nabla_\alpha f|(x) \leq \frac{c(|\alpha|, m)}{|x|^m}$ for any $m \in \mathbb{N}_0$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$: $\alpha_i \in \mathbb{N}_0$ for all $i = 1, \dots, n$, $|\alpha| := \alpha_1 + \alpha_2 + \dots \alpha_n$, and we have used the notation $\nabla_\alpha f := \nabla_{\alpha_1} \nabla_{\alpha_2} \dots \nabla_{\alpha_n} f$.

Using the represenation $b(x,t) = t^{-n/4} f(-\frac{x}{t^{1/4}})$ and the fact that f is in $\mathcal{S}$ we get

$$\begin{aligned}
\left| \left(\frac{\partial}{\partial t}\right)^l \left(\nabla^k\right) b(x,t) \right| &\le (t^{-p(k,l)} + t^{-1/4}) \sum_{0\le|\alpha|\le k+l} |\nabla_\alpha f| \left(-\frac{x}{t^{1/4}}\right) \\
&\le (t^{-p(k,l)} + t^{-1/4}) \frac{c(k,l,m)}{|x/t^{1/4}|^m} \\
&= c(k,l,m) \frac{t^{-p(k,l)+m/4} + t^{(m-1)/4}}{|x|^m}
\end{aligned}$$

which proves the estimate (6.1) since $m \in \mathbb{N}_0$ was arbitrary. This shows that the function $u(x,t) := \int_{\mathbb{R}^n} u_0(x-y) b(y,t) dy = \int_{\mathbb{R}^n} u_0(z) b(x-z,t) dz$ is well defined for any measurable L^∞ function $u_0 : \mathbb{R}^n \to \mathbb{R}$ for all $t > 0$, and is differentiable in time and space for all $t > 0$ for all $x \in \mathbb{R}^n$ and the derivative is given by differentiating under the integral sign (in view of the Lebesgue dominated convergence Theorem):

$$\left(\frac{\partial}{\partial t}\right)^l \left(\nabla^k\right) u(x,t) = \int_{\mathbb{R}^n} u_0(z) \left(\left(\frac{\partial}{\partial t}\right)^l \left(\nabla^k\right) b \right) (x-z,t) dz.$$

Using the fact that $\frac{\partial}{\partial t} b = -\Delta^2 b$ (see below for an explanation), we get $u : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}$ is smooth and satisfies $\frac{\partial}{\partial t} u = -\Delta^2 u$. Notice also that $\int_{\mathbb{R}^n} b(x,t) dx = \int_{\mathbb{R}^n} t^{-n/4} f(-\frac{x}{t^{1/4}}) dx = \int_{\mathbb{R}^n} f(-z) dz = \int_{\mathbb{R}^n} b(z,1) dz = 1$ (the last equality is explained below). Hence, for $x \in B_\varepsilon(z)$ where $z = (z_1, \ldots, z_n)$ has $z_1 > 2\varepsilon$, we have

$$\begin{aligned}
|u(x,t) - 1| &= \left| \int_{\mathbb{R}^n} b(x-y,t)(u_0(y) - 1) dy \right| \\
&= \left| \int_{B_\varepsilon(x)} b(x-y,t)(u_0(y)-1) dy \right. \qquad (6.2) \\
&\quad \left. + \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} b(x-y,t)(u_0(y)-1) dy \right| \\
&= 0 + \left| \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} b(x-y,t)(u_0(y)-1) dy \right| \\
&\le \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} 2|b(x-y,t)| dy \\
&\le \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{c(m,n) t^{m-n/4}}{|x-y|^{4m}} dy \\
&\le C(\varepsilon,m,n) t^{m-n/4} \\
&\le C(\varepsilon,m,n) t^2 \qquad (6.3)
\end{aligned}$$

for $m > 2 + n/4$ for $t \le 1$ which goes to zero as $t \to 0$. Similarly $|u(x,t)| \le C(\varepsilon,m,n) t^2$ goes to zero for all $x \in B_\varepsilon(z)$ where $z = (z_1, \ldots, z_n)$ has $z_1 < -2\varepsilon$. Hence $u(\cdot,t) \to u_0$ uniformly on compact sets $K \subseteq \mathbb{R}^n \backslash \{x \in \mathbb{R}^n \mid x_1 = 0\}$. The definition of u_0 and u guarantees that $u(cx, c^4 t) = u(x,t)$ for all $c, t > 0$. We verify this now. Notice first that

$$\begin{aligned} b(cx, c^4t) &= (2\pi)^{-n/2}(c^4t)^{-n/4}\int_{\mathbb{R}^n} e^{i(c^4t)^{-1/4}\langle w, cx\rangle - |w|^4}dw \\ &= c^{-n}(2\pi)^{-n/2}\int_{\mathbb{R}^n} e^{it^{-1/4}\langle w,x\rangle - |w|^4}dw \end{aligned} \tag{6.4}$$

that is $b(cx, c^4t) = c^{-n}b(x,t)$ for all $x \in \mathbb{R}^n$ for all $c > 0$. Also, the definition of u_0 guarantees that $u_0(cz) = u_0(z)$ for all $z \in \mathbb{R}^n$ and all $c > 0$. Making a change of variable $y = cw$ in the definition of u, and then using $b(cw, c^4t) = c^{-n}b(w,t)$ and the property of u_0 just mentioned, we calculate

$$\begin{aligned} u(cx, c^4t) := \int_{\mathbb{R}^n} u_0(cx - y)b(y, c^4t)dy &= \int_{\mathbb{R}^n} u_0(cx - cw)b(cw, c^4t)c^n dw \\ &= \int_{\mathbb{R}^n} u_0(x - w)b(w,t)dw \\ &= u(x,t). \end{aligned} \tag{6.5}$$

There must exist a point $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ with $\Delta u(x_0, t_0) \neq 0$: if not, then $\frac{\partial}{\partial t}u = -\Delta^2 u = 0$ for all $t > 0$, and hence $u(x,t) = u(x,s)$ for all $0 < s < t$, and hence , using $u \to u_0$ on $\mathbb{R}^n \backslash \{x \in \mathbb{R}^n | x_1 = 0\}$ as $t \searrow 0$ as explained above, we have $u(x,t) = 1$ on $\mathbb{R}^n \cap \{x \in \mathbb{R}^n | x_1 > 0\}$, $u(x,t) = 0$ on $\mathbb{R}^n \cap \{x \in \mathbb{R}^n | x_1 < 0\}$ for all $t > 0$, which contradicts the fact that $u(\cdot, t) : \mathbb{R}^n \to \mathbb{R}$ is smooth for $t > 0$.

So there exists $(x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}^+$ with $|\Delta u(x_0, t_0)|^2 \neq 0$. Now $u(cx, c^4, t) = u(x,t)$ for all $t > 0$, for all $x \in \mathbb{R}^n$ implies that (take the Laplacian w.r.t x of both sides) $c^2(\Delta u)(cx, c^4t) = (\Delta u)(x,t)$ which implies $|\Delta u|^2(cx, c^4t) = \frac{1}{c^4}|\Delta u|^2(x,t)$. In particular, choosing $t = t_0$, $c^4 = (s/t_0)$ and $x = x_0$ we find

$$|\Delta u|^2((s/t_0)^{1/4}x_0, s) = \frac{t_0}{s}|\Delta u|^2(x_0, t_0)$$

and hence $|\Delta u|^2(x_s, s) = \frac{k_0}{s}$ where $k_0 = t_0|\Delta u(x_0,t_0)|^2 \neq 0$ and $x_s = (s/t_0)^{1/4}x_0$. Using an almost identical argument, we see that for all $t > 0$, there must be points $y(t) \in \mathbb{R}^n$ such that $(\Delta^2 u)(y(t), t) = \frac{k_1}{t}$ for some fixed $k_1 \in \mathbb{R}$, $k_1 \neq 0$.

The fact that $\frac{\partial}{\partial t}b = -\Delta^2 b$ can be seen as follows. Using Theorem 1.7 of Chapter I.1 in [SW], we have $-|x|^4 e^{-|x|^4t} = \frac{\partial}{\partial t}(e^{-|x|^4t}) = (\frac{\partial}{\partial t}\widehat{b})(x,t) = \widehat{(\frac{\partial}{\partial t}b)}(x,t) = \widehat{(-\Delta^2 b)}(x,t)$, and hence, taking the inverse of the Fourier transform, we get $(\frac{\partial}{\partial t}b) = -\Delta^2 b$ (note that $(\frac{\partial}{\partial t}\widehat{b})(x,t) = \widehat{(\frac{\partial}{\partial t}b)}(x,t)$ is true in view of the Lebesgue dominated convergence Theorem and the estimates (6.1) and the inverse of the Fourier transform exists in view of Corollary I.21 in I.1 of [SW] and the fact that b is in $\mathcal{S}$). The fact that $\int_{\mathbb{R}^n} b(z,1)dz = 1$ may be seen by looking at how b was derived: Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be a smooth function which is equal to 1 on $B_1(0)$ and has compact support on $B_2(0)$. Hence u_0 is in $\mathcal{S}$, and the Fourier transform $\widehat{u_0}$ of u_0 is also in $\mathcal{S}$. We only take the Fourier transform in the space direction in that which follows. Write $u(x,t) = (b(\cdot,t) * u_0)(x)$ so $\frac{\partial}{\partial t}u = -\Delta^2 u$ as explained above, and

$$\widehat{u}(x,t) = \widehat{b}(x,t)\widehat{u_0}(x) = e^{-t|x|^4}\widehat{u}_0(x) \tag{6.6}$$

(see Theorem 1.4 in I.1 os [SW]) and hence $\widehat{u}(\cdot,t) \to \widehat{u_0}(\cdot)$ in the L^2 sense as $t \searrow 0$. But this implies $u(\cdot,t) \to u_0(\cdot)$ in the L^2 sense as $t \searrow 0$, in view of the fact that the L^2 norm is preserved for the Fourier transform (and the inverse Fourier transform) of a function in $L^2 \cap L^1$ (see Theorem 2.1 and Theorem 2.4 in Chapter I.2 of [SW]).

In particular this shows $\int_{\mathbb{R}^n} b = 1$: If $1 \neq c_0 := \int_{\mathbb{R}^n} b \neq 0$, then for $x \in B_{1/2}(0)$ we have

$$\begin{aligned}
|u(x,t) - 1/c_0| &= \left| \int_{\mathbb{R}^n} b(x-y,t)(u_0(y)-1)dy \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_1(0)} b(x-y,t)(u_0(y)-1)dy \right| \\
&= \left| \int_{\mathbb{R}^n \setminus B_1(0)} b(x-y,t)(u_0(y)-1)dy \right| \\
&\leq C \int_{\mathbb{R}^n \setminus B_1(0)} |b(x-y,t)|dy \\
&= C \int_{\mathbb{R}^n \setminus B_1(x)} |b(z,t)|dz \\
&\leq C \int_{\mathbb{R}^n \setminus B_{1/2}(0)} |b(z,t)|dz \\
&\to 0 \qquad (6.7)
\end{aligned}$$

as $t \searrow 0$ in view of (6.1), which shows $u(\cdot,t)$ converges uniformly in the supremum norm to $(1/c_0) \neq 1$ on $B_{1/2}(0)$ as $t \searrow 0$, which contradicts the fact that $u(\cdot,t)$ converges to u_0 in the L^2 norm as $t \searrow 0$. Similarly, if $\int_{\mathbb{R}^n} b = 0$, one shows $u(x,t) \to 0$ uniformly in the supremum norm on $B_{1/2}(0)$ as $t \searrow 0$, which contradicts the fact that $u(\cdot,t)$ converges to u_0 in the L^2 norm as $t \searrow 0$.

7. Appendix

We require a rather specific form of the standard interpolation inequalities (see for example [GT, Theorems 7.25–7.28]). We have thus provided proofs and precise statements of that which we need here in the appendix.

Lemma 7.1. *For any smooth function $\varphi : B_1(0) \to \mathbb{R}$ we have*

$$\int_{B_{1-s\sigma}} |\nabla^{2s}\varphi|^2 + |\nabla^{2s-1}\varphi|^2 \leq c(s,\sigma) \int_{B_1} |\Delta^s \varphi|^2 + |\Delta^{s-1}\varphi|^2 + \ldots + |\varphi|^2 \tag{7.1}$$

for any $s \in \mathbb{N}$ and $1 > \sigma > 0$, as long as $1 - s\sigma > 0$.

Proof. We show the inequality (7.1) for arbitrary smooth $\varphi : B_1(0) \to \mathbb{R}$ using induction.

Step 1: L^2-theory (see for example [E, Theorem 1, Section 6.3.1]) tells us that for an arbitrary smooth function $\varphi : B_1 \to \mathbb{R}$

$$\int_{B_{1-\sigma}} |\nabla^2\varphi|^2 + |\nabla\varphi|^2 \leq c(\sigma) \int_{B_1} |\Delta\varphi|^2 + |\varphi|^2 \tag{7.2}$$

as required.

Inductive Step: Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a multi-index with $0 \leq \alpha_i \leq 2s$ and $\sum_{i=1}^n \alpha_i = 2s$. We use the notation $\nabla_\alpha \varphi := \nabla_{\alpha_1}\nabla_{\alpha_2} \ldots \nabla_{\alpha_n}\varphi$. Assume that statement (7.1) is true for some $s \in \mathbb{N}$. Again, L^2 theory, see for example [E] Theorem

1, Section 6.3.1, tells us that

$$\begin{aligned}
\int_{B_{1-(s+1)\sigma}} & |\nabla^2\nabla_\alpha\varphi|^2 + |\nabla\nabla_\alpha\varphi|^2 \\
&\le a(s,\sigma)\int_{B_{1-s\sigma}} |\Delta(\nabla_\alpha\varphi)|^2 + |\nabla_\alpha\varphi|^2 \\
&= a(s,\sigma)\int_{B_{1-s\sigma}} |\nabla_\alpha(\Delta\varphi)|^2 + |\nabla_\alpha\varphi|^2 \\
&\le a(s,\sigma)c(s,\sigma)\int_{B_1} |\Delta^s(\Delta\varphi)|^2 + |\Delta^{s-1}(\Delta\varphi)|^2 + \ldots + |\Delta\varphi|^2 \\
&\quad + a(s,\sigma)c(s,\sigma)\int_{B_1} |\Delta^s\varphi|^2 + |\Delta^{s-1}\varphi|^2 + \ldots + |\varphi|^2, \\
&= \tilde{a}(s,\sigma)\int_{B_1} |\Delta^{s+1}\varphi|^2 + |\Delta^s\varphi|^2 + \ldots + |\varphi|^2
\end{aligned}$$

where in the second last line (the inequality) we have used the induction hypothesis applied to the functions $\Delta\varphi$ and φ. By summing up over all possible α (the number of possible α is a constant depending on n and s) we see that

$$\int_{B_{1-(s+1)\sigma}} |\nabla^{2s+2}\varphi|^2 + |\nabla^{2s+1}\varphi|^2 \le c(s+1,\sigma)\int_{B_1} |\Delta^{s+1}\varphi|^2 + |\Delta^s\varphi|^2 + \ldots + |\varphi|^2$$

as required.

This completes the proof by induction. □

Lemma 7.2. *Suppose $u \in C^\infty_{loc}(\mathbb{R}^n)$, $k \in \mathbb{N}$, $s > 8$, and γ, c_γ are as in (γ). For any $\delta_0 > 0$ we have*

$$\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \le \delta_0\rho^4\int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{c_{\delta_0}}{\rho^4}\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,.$$

where c_{δ_0} is an absolute constant given by

$$c_{\delta_0} = c_{\delta_0}(\delta_0, s, n) = 2^4(\delta_0^{-1} + 2^9 s^4 c_\gamma^4)\,.$$

Proof. Integrating by parts,

$$
\begin{aligned}
\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} &= -\int_{\mathbb{R}^n} (\nabla_i \Delta^k u)(\nabla_i \Delta^{k-1} u)\gamma^{s-4} \\
&\qquad - (s-4)\int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k-1} u)(\nabla_i \gamma)\gamma^{s-5} \\
&= \int_{\mathbb{R}^n} (\Delta^{k+1} u)(\Delta^{k-1} u)\gamma^{s-4} \\
&\qquad + (s-4)\int_{\mathbb{R}^n} (\nabla_i \Delta^k u)(\Delta^{k-1} u)(\nabla_i \gamma)\gamma^{s-5} \\
&\qquad - (s-4)\int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k-1} u)(\nabla_i \gamma)\gamma^{s-5} \\
&= \int_{\mathbb{R}^n} (\Delta^{k+1} u)(\Delta^{k-1} u)\gamma^{s-4} \\
&\qquad - 2(s-4)\int_{\mathbb{R}^n} (\Delta^k u)(\nabla_i \Delta^{k-1} u)(\nabla_i \gamma)\gamma^{s-5} \\
&\qquad - (s-4)\int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k-1} u)(\Delta \gamma)\gamma^{s-5} \\
&\qquad - (s-4)(s-5)\int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k-1} u)|\nabla \gamma|^2\gamma^{s-6} \,.
\end{aligned}
$$

Estimating the right hand side with Young's inequality (estimate (2.3)), we have for any $\delta_i > 0$

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\quad \le \delta_1 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{1}{4\delta_1 \rho^4}\int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \\
&\qquad + \delta_2 \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c_\gamma^2 (s-4)^2}{\delta_2 \rho^2}\int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \\
&\qquad + \delta_3 \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c_\gamma^2 (s-4)^2}{4\delta_3 \rho^4}\int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-6} \\
&\qquad + \delta_4 \int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} + \frac{c_\gamma^4 (s-4)^2 (s-5)^2}{4\delta_4 \rho^4}\int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \,,
\end{aligned}
$$

which upon absorption gives

$$
\begin{aligned}
&\int_{\mathbb{R}^n} |\Delta^k u|^2 \gamma^{s-4} \\
&\quad \le 4\delta_1 \rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1} u|^2 \gamma^s + \frac{16 c_\gamma^2 (s-4)^2}{\rho^2}\int_{\mathbb{R}^n} |\nabla \Delta^{k-1} u|^2 \gamma^{s-6} \\
&\qquad + \frac{1}{\rho^4}\left(\frac{1}{\delta_1} + 4c_\gamma^2 (s-4)^2 + 4c_\gamma^4 (s-4)^2 (s-5)^2\right)\int_{\mathbb{R}^n} |\Delta^{k-1} u|^2 \gamma^{s-8} \,,
\end{aligned}
\tag{7.3}
$$

where we have chosen $\delta_2 = \delta_3 = \delta_4 = \frac{1}{4}$, and we used $\gamma^{s-6}(\cdot) \le \gamma^{s-8}(\cdot)$, which follows in view of $0 \le \gamma \le 1$ and $s-6 > s-8 \ge 1$. For the second term we integrate by parts and estimate using Young's inequality to obtain

$$\begin{aligned}
&\int_{\mathbb{R}^n} |\nabla\Delta^{k-1}u|^2\gamma^{s-6} \\
&\quad = -\int_{\mathbb{R}^n} (\Delta^k u)(\Delta^{k-1}u)\gamma^{s-6} - (s-6)\int_{\mathbb{R}^n} (\Delta^{k-1}u)(\nabla_i\Delta^{k-1}u)(\nabla_i\gamma)\gamma^{s-7} \\
&\quad \le \frac{\rho^2}{64c_\gamma^2(s-4)^2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} + \frac{16c_\gamma^2(s-4)^2}{\rho^2}\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8} \\
&\qquad + \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\Delta^{k-1}u|^2\gamma^{s-6} + \frac{c_\gamma^2(s-6)^2}{2\rho^2}\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,.
\end{aligned}$$

Absorbing the third term on the right into the left yields

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla\Delta^{k-1}u|^2\gamma^{s-6} &\le \frac{\rho^2}{32c_\gamma^2(s-4)^2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \\
&\quad + \frac{c_\gamma^2}{\rho^2}\Big(32(s-4)^2 + (s-6)^2\Big)\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,,
\end{aligned}$$

and so

$$\begin{aligned}
\frac{16c_\gamma^2(s-4)^2}{\rho^2}\int_{\mathbb{R}^n} |\nabla\Delta^{k-1}u|^2\gamma^{s-6} &\le \frac{1}{2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \\
&\quad + \frac{16c_\gamma^4(s-4)^2}{\rho^4}\Big(32(s-4)^2 + (s-6)^2\Big)\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,.
\end{aligned}$$

Combining the above with (7.3) we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} &\le 4\delta_1\rho^4\int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{1}{2}\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \\
&\quad + \frac{1}{\rho^4}\bigg(16c_\gamma^4(s-4)^2\Big(32(s-4)^2 + (s-6)^2\Big) \\
&\qquad + \frac{1}{\delta_1} + 4c_\gamma^2(s-4)^2 + 4c_\gamma^4(s-4)^2(s-5)^2\bigg)\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}\,,
\end{aligned}$$

Absorbing the second term on the right into the left, multiplying through by 2 and choosing $\delta_1 = \frac{\delta_0}{8}$ yields the estimate

$$\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \le \delta_0\rho^4\int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{\tilde{c}_{\delta_0}}{\rho^4}\int_{\mathbb{R}^n} |\Delta^{k-1}u|^2\gamma^{s-8}$$

where

$$\begin{aligned}
\tilde{c}_{\delta_0} &= 32c_\gamma^4(s-4)^2\left(32(s-4)^2 + (s-6)^2\right) + \frac{16}{\delta_0} + 8(s-4)^2\left(c_\gamma^2 + c_\gamma^4(s-5)^2\right) \\
&= \frac{16}{\delta_0} + 8(s-4)^2c_\gamma^2\left(1 + c_\gamma^2(s-5)^2 + 4c_\gamma^2\left(32(s-4)^2 + (s-6)^2\right)\right)\,.
\end{aligned}$$

Since $s > 8$ and $c_\gamma \geq 1$ we have

$$\begin{aligned}\tilde{c}_{\delta_0} &\leq \frac{16}{\delta_0} + 8s^2c_\gamma^2\left(1 + c_\gamma^2 s^2 + 4c_\gamma^2\left(32s^2 + s^2\right)\right)\\ &\leq \frac{2^4}{\delta_0} + 2^3 s^4 c_\gamma^2\left(1 + c_\gamma^2 + 2^2 c_\gamma^2\left(2^5 + 1\right)\right)\\ &\leq 2^4(\delta_0^{-1} + 2^9 s^4 c_\gamma^4)\\ &:= c_{\delta_0}\,,\end{aligned}$$

yielding the constant claimed. □

Corollary 7.3. *Suppose $u \in C^\infty_{loc}(\mathbb{R}^n)$, $k \in \mathbb{N}$, and γ, c_γ are as in (γ). For all $k \in \mathbb{N}$, $s > 4k$,*

$$\int_{\mathbb{R}^n} |\Delta^k u|^2\gamma^{s-4} \leq \delta\rho^4 \int_{\mathbb{R}^n} |\Delta^{k+1}u|^2\gamma^s + \frac{c}{\rho^{4k-4}}\int_{\mathbb{R}^n}|\Delta u|^2\gamma^{s-4k}\,,$$

where $c(\delta, s, n) < \infty$ is a constant depending on δ, s, n.

Proof. We shall proceed by induction in $k \in \mathbb{N}$. We wish to show that

$$E^k_{\gamma^{s-4}}(u) \leq \hat{\delta}\rho^4 E^{k+1}_{\gamma^s}(u) + \frac{\hat{c}_{\hat{\delta},k,s}}{\rho^{4k-4}} E^1_{\gamma^{s-4k}}(u)\,. \tag{7.4}$$

is true for all $s > 4k$ for all $\hat{\delta} > 0$, where $\hat{c}_{\hat{\delta},k,s} = \hat{c}(\hat{\delta}, s, k, n)$. Let $k = 1$. Then (7.4) is true for all $s > 4k$ for all $\delta > 0$ trivially with the choice of $\hat{c}_{\hat{\delta},k,s} = 1$.

Assume (7.4) for some fixed $k \in \mathbb{N}$, and let $s > 4(k+1)$,$\delta > 0$ be arbitrary. Then $s > 8$ and Lemma 7.2 gives the estimate

$$E^l_{\gamma^{s-4}}(u) \leq \delta\rho^4 E^{l+1}_{\gamma^s}(u) + \frac{c_{\delta,l,s}}{\rho^4}E^{l-1}_{\gamma^{s-8}}(u) \tag{7.5}$$

for any $l \in \mathbb{N}$, where $c_{\delta,l,s} = c(\delta, l, s, n)$.

Using this inequality with $l = k+1 \in \mathbb{N}$ and then (7.4) we have

$$\begin{aligned}E^{k+1}_{\gamma^{s-4}}(u) &\leq \delta\rho^4 E^{k+2}_{\gamma^s}(u) + \frac{c_{\delta,k+1,s}}{\rho^4}E^k_{\gamma^{s-8}}(u)\\ &\leq \delta\rho^4 E^{k+2}_{\gamma^s}(u) + \hat{\delta}c_{\delta,k+1,s}E^{k+1}_{\gamma^{s-4}}(u) + \frac{\hat{c}_{\hat{\delta},k,s-4}c_{\delta,k+1,s}}{\rho^{4k}}E^1_{\gamma^{s-4-4k}}(u)\,.\end{aligned}$$

where here we used the fact that $s > 4(k+1) = 4k+4$ implies that $\tilde{s} = s - 4 > 4k$ and so (7.4) is valid with s replaced by $\tilde{s}$. Choosing $\hat{\delta}c_{\delta,k+1,s} = \frac{1}{2}$ and absorbing we obtain

$$E^{k+1}_{\gamma^{s-4}}(u) \leq 2\delta\rho^4 E^{k+2}_{\gamma^s}(u) + 2\frac{\hat{c}_{\hat{\delta},k,s-4}c_{\delta,k+1,s}}{\rho^{4k}}E^1_{\gamma^{s-4-4k}}(u)\,.$$

which gives us the result, as $\delta > 0$ was arbitrary. Note that we can ensure the constant only depends on n, s and not k by taking the supremum of the constants we obtained in this argument, where this supremum is taken over all $4k < s$, $k \in \mathbb{N}$, for a fixed $s \in \mathbb{N}$. □

Acknowledgements

The first author would like to thank the University of Wollongong, where part of this work was carried out. The second author was partially supported by Alexander-von-Humboldt fellowship 1137814 at the Otto-von-Guericke Universität Magdeburg and by Australian Research Council Discovery Project grant DP120100097 at the University of Wollongong. Further partial support toward both authors was provided by University of Wollongong Research Council 2014 Small Grant 228381024. They are grateful for their support.

References

[Ar] D. G Aronson, *Non-negative solutions of linear parabolic equations*, Ann. Scuola Norm. Sup. Pisa Cl. Sei. (4) 22 (1968), 607-694.

[E] L. C. Evans, *Partial differential equations*, Graduate studies in Mathematics, American Mathematical Society, 1998.

[GP] V.A. V.A. Galaktionov, S.I. Pohožaev, Existence and blow-up for higher-order semilinear parabolic equations: Majorizing order-preserving operators, Indiana Univ. Math. J. 51 (2002) 1321-1338.

[GG1] Gazzola, F., Grunau, H.-C. *Some new properties of biharmonic heat kernels* Nonlinear analysis 70 (2009) pages. 2965-2973

[GG2] Gazzola, F., Grunau, H.-C. *Eventual local positivity for a biharmonic heat equation in* $\mathbb{R}^n$, Discrete Contion.Syst. S 1 (2008) 83-87

[GT] D. Gilbarg, and N. S. Trudinger, *Elliptic partial differential equations of second order*. Vol. 224. Springer Verlag, 2001.

[J] F. John, *Partial Differential Equations*. Springer Verlag, 1982.

[KL] L. Karp and P. Li, *The heat equation on complete Riemannian manifolds*, unpublished paper, available at http://www.math.uci.edu/ pli/

[Si] M. Simon, *Local Results for Flows Whose Speed or Height Is Bounded by c/t*, Int. Math. Res. Not. IMRN, Vol. 2008, (2008), 14 pages.

[SW] E. Stein, G. Weiss, *Introduction to Fourier analysis*, Princeton University Press, (1971).

[T] A. Tychonoff, *Theóremes d'unicité pour l'équation de la chaleur*, Mat. Sb. 42:2 (1935), 199–216.

[W] D. V. Widder, *Positive temperatures on an infinite rod*, Trans. Amer. Math. Soc. 55 (1944), 85-95.

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